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Symmetry of the Matrix Model of Anisotropic Media with Orthogonal Eigenmodes and its Application for the Developing of Remote Sensing Polarimetric Measurement Systems in Visible

Sergey N. Savenkov

National Taras Shevchenko University of Kiyv
Department of Quantum Radiophysics
64, Volodymyrska Street
Kiev, Ukraine 01601

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This report results from a contract tasking National Taras Shevchenko University of Kiyv as follows: This project studied theoretically and experimentally the Mueller matrix model of crystalline anisotropic medium basing on generalized polarimetric equivalence theorem. Knowledge of these features is important for experimental retrieving the materials and media parameters and for classification of various types of media on the basis of their scattering models. Main objective of this study was determination of the conditions under which crystalline anisotropic medium has orthogonal eigenpolarizations (eigenmodes). It is known all four basic types of anisotropy, circular and linear birefringence and circular and linear dichroism, each taken separately, possess orthogonal eigenpolarizations. Generalized birefringence, i.e. the case of medium simultaneously exhibiting linear and circular birefringence is characterized by unitary matrix model and has orthogonal eigenpolarizations. At the same time, simultaneous presence of dichroism and birefringence in a medium may lead to nonorthogonal eigenpolarizations. So a systematic study of conditions under which such medium may possesses orthogonal eigenpolarizations was proposed.
Initial expected results of this research includes: (i) ascertainment of generalized conditions for orthogonality of homogeneous anisotropic medium eigenmodes; (ii) determination of structure and symmetry of matrix model for such class of media; (iii) development of a polarimeter for the measurement of determined structures of incomplete Mueller matrices; (iv) assembling of the breadboard of 1D polarimeter; (v) check out of breadboard of 1D polarimeter.

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Symmetry of the Matrix Model of Anisotropic Media with Orthogonal Eigenmodes and its Application for the Developing of Remote Sensing Polarimetric Measurement Systems in Visible

Technical Report

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1 November 2010 – 1 November 2011

National Taras Shevchenko University of Kiyv
Department of Quantum Radiophysics
Kiev, Ukraine

Sergey N. Savenkiov

Contact telephone numbers: +38 044 526-05-80
Contact email: sns@univ.kiev.ua

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1. Introduction

One of the most promising approach for remote sensing is polarimetry. Polarimetry assumes the exposure of the medium with electromagnetic radiation with a given polarization and the subsequent study of the transformation of this polarization as a result of interaction between incident radiation and studied medium.

It is known that if the interaction of radiation with the medium is linear, then it mathematically can be represented by a system of linear equations:

$$\mathbf{E}^{\mathbf{out}} = \mathbf{T}\,\mathbf{E}^{\mathbf{in}} \tag{1}$$

where $\mathbf{E}^{\mathbf{in(out)}}$ - vectors representing the polarization of input and output radiation; $\mathbf{T}$ - is a matrix that defines a linear transformation of polarization state of input radiation into the output one.

When one intends to find the unknown parameters of the output radiation $\mathbf{E}^{\mathbf{in(out)}}$ for a given $\mathbf{T}$, this is the so-called "direct problem". We interested in the "inverse problem", the essence of which is finding the explicit form of the matrix $\mathbf{T}$. It is clear that the matrix $\mathbf{T}$ contains all information about the properties of the medium in implicit form. To extract this information a number of relevant matrix models have been developed. The most perspective are the models based on the singular and the polar decompositions [1-5]. Using the singular and the polar decomposition with respect to the operator $\mathbf{T}$ gives the possibility to represent an investigated media as a finite set of layers with well-studied and easily interpretable anisotropic properties. In the case of singular value decomposition the medium is represented as a set of four layers – one with a linear dichroism, one with the optical activity, and two layers with linear birefringence [6]. By polar decomposition the same medium is represented as two layers, which are the elliptical polarizer and an elliptical phase plate of general forms [2-5]. Thus, the complex anisotropy of the arbitrary deterministic medium is decomposed into simple components describing separately the transformation of phase and amplitude of the input radiation by studied medium.

The fact that the model basing on singular value decomposition contains two components with the same types of anisotropy (linear birefringence) complicates the unique interpretation of decomposition results and thus leads to a relatively rare using of this model. Polar decomposition is used much more frequently, see for example Refs. 4,5,7 and 9. However, from our point of view, the parameters which are obtained in polar decomposition, i.e., diatenuation vector, polarizance and retardation, have also no clear physical interpretation.

In scope of this project we used a model based on the so-called generalized equivalence theorem [10]. In concordance with this theorem arbitrary deterministic medium can be presented as a product of four basic types of anisotropy: linear phase and amplitude and circular phase and amplitude. Physics of these types of anisotropy is studied well and parameters characterizing these type of anisotropy have clear.

Note that all mentioned multiplicative layered models of anisotropic media consist of the components which characterized by orthogonal eigenpolarizations. However the eigenpolarizations of the initial medium are not necessarily orthogonal. Elliptical polarizer in polar decomposition also has orthogonal eigenpolarizations. At the same time, the medium represented by a sequence of layers with dichotic properties: linear and circular dichroism, in general has a non-orthogonal eigenpolarizations. The absence of dichroism in medium results immediately in orthogonality of eigenpolarizations.

The cases of orthogonal eigenpolarizations usually receives increased attention in the literature, and it has been a milestone for classifying polarization elements and studying their properties [1,2,6,11]. However, in scope of multiplicative matrix models there is no systematic study of the conditions under which the arbitrary anisotropic medium will have orthogonal eigenpolarizations in general case. Derivation of these conditions will determine the structure and symmetry of the matrix model of corresponding class of media.

Thus, the main goal of this study is derivation of the conditions, under which arbitrary crystalline anisotropic medium has in general the orthogonal eigenpolarizations. In addition we (i) developed the scheme of polarimeter, which is optimally fit for the measurement of structures of Mueller matrices with orthogonal eigenpolarizations; and (ii) assembled the breadboard of 1D polarimeter, which meets the requirements derived in scope of this research.

Because of unknown structure of the matrix model for media with orthogonal eigenpolarizations in general case developed polarimeter should provide an “operation flexibility” between various measurement strategies for optimal measurement of given structures of matrices. One way to do this is using in input and output channels of polarimeter the polarization transducers with universally controllable parameters.

Currently a number of schemes to measure the Stokes parameters of electromagnetic radiation (Stokes-polarimeter) have been proposed and implemented [11-15]. Key elements for these systems are polarization transducers with controllable parameters. Basing on approach to control of polarization parameters the Stokes-polarimeters can be divided into two classes: mechanically controllable and electrically controllable. Mechanically controlled transducers are the ones which parameters are mechanically altered (moved or rotated polarizers, phase plates, etc.). Schemes of Stokes-polarimeter with mechanically controlled polarization transducers are widely-spread due to simplicity in implementation and adjusting. However, they are characterized by significant disadvantages due to impossibility of providing the necessary level of accuracy and performance.

Among the electrically controlled transducers it can be pointed out electro-, magneto-, acousto-optical cells etc. Anisotropic properties of these transducers are changed by applying an external electric (magnetic) field which can be accomplished with a quite high speed and accuracy without mechanical moving. Disadvantages of electrically controlled polarization transducers are: 1) nonlinear dependence of parametres on the applied external field, and 2) limitation of the range of anisotropy values, 3) as a rule only one parameter can be controlled (the orientation axis of anisotropy, values of anisotropy etc.). These disadvantages somewhat complicate the adjustment of Stokes-polarimeter and require the using of more than one transducer, or changing their orientations to expand the range of polarization transformation.
Recently transducers on liquid crystals (LC) cells [16-29] become popular. LC cell consists of a layer of liquid crystal located between two transparent electrodes controlled by external voltage. This cell exhibits birefringence which axis is normal to the propagation direction. Thus, the LC cell is a wave plate (LC retarder) which phase shift (birefingence) is controlled by voltage. Coupling the relatively low price and satisfactory performance the LC cell allows to get a wide range of changes of birefringence by controlling voltage near 15-30V. Two consequently placed LC cells provide a range of polarization modulation of radiation enables to determine all four Stokes parameters.

2. Methods, Assumptions and Procedures

2.1 Jones and Mueller matrix methods. Spectral problem

To describe the linear interaction of polarized radiation with the medium, see Eq.(1), the Jones and Mueller matrix methods, which is uniquely related in case of a homogeneous anisotropic media [30], are used.

When Jones matrix method is used than in Eq.(1) $\mathbf{E}^{\mathbf{in(out)}}$ is the Jones vector of input (output) radiation, and $\mathbf{T}$ denotes the Jones matrix.

Jones matrix **T** (2x2 matrix with complex elements t_{mn}) describes anisotropic properties of homogeneous medium:

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}; \quad \text{where } t_{mn} = |t_{mn}| \exp(-i\phi_{mn}). \tag{2}$$

Eigenpolarizations χ of such matrix can be obtained as:

$$\chi_{1,2} = \frac{1}{2}\frac{t_{22} - t_{11} \pm \sqrt{(t_{22} - t_{11})^2 + 4t_{12} \cdot t_{21}}}{t_{21}}; \tag{3}$$

where $\chi = E_x / E_y$ - complex variable [30]; $E_{x,y}$ - components of the Jones vector $\mathbf{E}$.

For orthogonal eigenpolarizations the following relation:

$$\chi_1 \chi_2^* = -1 \tag{4}$$

is satisfied.

In experimental studies the Mueller method is used for description of interaction between electromagnetic radiation and medium because it operates with intensities of radiation that can be directly measured.

When, in scope of the Muller matrix method Eq.(1) can be rewritten as:

$$\mathbf{S}^{\mathbf{out}} = \mathbf{M}\,\mathbf{S}^{\mathbf{in}} \tag{5}$$

where $\mathbf{S}^{\mathbf{in(out)}}$ - denotes input (output) Stokes vector.

The definition of Stokes vector is follows :

$$\mathbf{S} = \begin{bmatrix} I \\ Ip\cos(2\beta)\cos(2\varepsilon) \\ Ip\sin(2\beta)\cos(2\varepsilon) \\ Ip\sin(2\varepsilon) \end{bmatrix}, \tag{6}$$

where I – overall intensity of radiation; p – polarization degree; ϑ - azimuth and ε - ellipticity angle of polarization ellipse.

It can be seen from Eq.(6) that the elements of Stokes vector (called the Stokes parameters) have dimensions of intensity. Thus, the Stokes parameters and Mueller matrix elements can be measured directly in experiment. Besides, the Stokes parameters describe either completely polarized (p=1), partially polarized (0<p≤1) and depolarized (p=0) radiation. Similarly, the Mueller matrix can represent both homogeneous and inhomogeneous media.

In accordance with Eq.(5) the Mueller matrix **M** is a 4x4 matrix with real elements

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}. \tag{7}$$

This matrix, as well as Jones matrix, describes completely anisotropic properties of homogeneous medium for a given input and output (scattering) directions and wavelength of input radiation.

Direct solving of the spectral problem [31] in scope of Mueller formalism, i.e., finding the conditions on Mueller matrix elements for eigenpolarizations to be orthogonal, is quite complicated task because of Mueller matrix dimension. However, this problem, as it is demonstrated below, can be solved in scope of the Jones formalism both in terms of matrix elements and in terms of anisotropy parameters (introduced below, see subsection 2.3). Taking into account the fact that Mueller **M** and Jones **T** matrices for homogeneous medium are interconnected by relation

$$\mathbf{M} = \mathbf{A}(\mathbf{T} \otimes \mathbf{T}^*)\mathbf{A}^{-1}, \tag{8}$$

where $*$ - conjugation; $\otimes$ - Kronecker product; and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix},$$

all results, which are obtained for the Jones formalism, can be translated to the Mueller formalism with Eq.(8). Note that the main condition for that is the medium under consideration does not depolarize input radiation.

It is important to note that any 2x2 matrix could be called "Jones matrix", i.e., arbitrary 2x2 matrix with complex elements describes always a physical realizable transformation of polarization trough the Eq.(1). The same can not to be said about any 4x4 matrix with real elements. To be named as "Mueller matrix" this matrix has to meet an ample of requirements [32].

Due to above conditions the solving of spectral problem even for Mueller-Jones matrix (which is determined by Eq.(8)) case is generally difficult. We can write it in the following form:

$$\mathbf{M}\left(\nu_1^{-1}\mathbf{S}_1 + \nu_2^{-1}\mathbf{S}_2\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

2.2 Generalized equivalence theorem

For analysis of anisotropic parameters of medium we use the so-called generalized matrix equivalence theorem [33]. According to this theorem an arbitrary Jones (Mueller-Jones) matrix of crystalline anisotropic medium can be represented as the product of the Jones matrices of four basic types of anisotropy: linear dichroism (linear amplitude anisotropy) $\mathbf{T}^{LA}$, circular dichroism (circular amplitude anisotropy) $\mathbf{T}^{CA}$, linear birefringence (linear phase anisotropy) $\mathbf{T}^{LP}$, and optical activity or circular birefringence (circular phase anisotropy) $\mathbf{T}^{CP}$:

$$\mathbf{T} = \mathbf{T}^{CP}\mathbf{T}^{lP}\mathbf{T}^{CA}\mathbf{T}^{LA}. \tag{10}$$

The Jones and Mueller matrices of basic types of anisotropy are well-known [30] and presented below.

Jones matrices:

$$\begin{aligned}
\mathbf{T}^{LA} &= \begin{bmatrix} \cos^2(\gamma) + \sin^2(\gamma)P & \cos(\gamma)\sin(\gamma)(1-P) \\ \cos(\gamma)\sin(\gamma)(1-P) & \sin^2(\gamma) + \cos^2(\gamma)P \end{bmatrix}; \\
\mathbf{T}^{CA} &= \begin{bmatrix} 1 & -i \cdot R \\ i \cdot R & 1 \end{bmatrix}; \\
\mathbf{T}^{LP} &= \begin{bmatrix} \cos^2(\alpha) + \sin^2(\alpha)e^{-i\Delta} & \cos(\alpha)\sin(\alpha)\left(1-e^{-i\Delta}\right) \\ \cos(\alpha)\sin(\alpha)\left(1-e^{-i\Delta}\right) & \sin^2(\alpha) + \cos^2(\alpha)e^{-i\Delta} \end{bmatrix}; \\
\mathbf{T}^{CP} &= \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}.
\end{aligned} \tag{11}$$

Mueller matrices:

$$\mathbf{M}^{CA}=\begin{bmatrix}1+R^2 & 0 & 0 & \pm 2R\\ 0 & 1-R^2 & 0 & 0\\ 0 & 0 & 1-R^2 & 0\\ \pm 2R & 0 & 0 & 1+R^2\end{bmatrix};$$

$$\mathbf{M}^{CP}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(2\varphi) & \sin(2\varphi) & 0\\ 0 & -\sin(2\varphi) & \cos(2\varphi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix};$$

(12)

$$\mathbf{M}^{LA}=\begin{bmatrix}1+P^2 & (1-P^2)\cos(2\gamma) & (1-P^2)\sin(2\gamma) & 0\\ (1-P^2)\cos(2\gamma) & \cos^2(2\gamma)(1+P^2)+2\sin^2(2\gamma)P & \cos(2\gamma)\sin(2\gamma)(1-P)^2 & 0\\ (1-P^2)\sin(2\gamma) & \cos(2\gamma)\sin(2\gamma)(1-P)^2 & \sin^2(2\gamma)(1+P^2)+2\cos^2(2\gamma)P & 0\\ 0 & 0 & 0 & 2P\end{bmatrix};$$

$$\mathbf{M}^{lP}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos^2(2\alpha)+\sin^2(2\alpha)\cos(\Delta) & \cos(2\alpha)\sin(2\alpha)(1-\cos(\Delta)) & -\sin(2\alpha)\sin(\Delta)\\ 0 & \cos(2\alpha)\sin(2\alpha)(1-\cos(\Delta)) & \sin^2(2\alpha)+\cos^2(2\alpha)\cos(\Delta) & \cos(2\alpha)\sin(\Delta)\\ 0 & \sin(2\alpha)\sin(\Delta) & -\cos(2\alpha)\sin(\Delta) & \cos(\Delta)\end{bmatrix},$$

where: value of linear dichroism is in range $P\in[0;1]$ and the azimuth of maximum transition is in range $\gamma\in[-\pi/2;\pi/2]$; value of circular dichroism is in range $R\in[-1;1]$; value of linear birefringence is in range $\Delta\in[0;2\pi]$ with it's fast axis orientation in range $\alpha\in[-\pi/2;\pi/2]$; value of optical activity is in range $\varphi\in[-\pi;\pi]$.

Parameters, which characterize basic types of anisotropy, have clear physical meanings. They associate with time and spatial non-locality of the medium response on input light, i.e., with time and spatial dispersion. Last ones determine the form of Maxwell's constitutive relations. Thus, generalized polarimetric equivalence theorem combines both mathematical generality and physical interpretability.

2.3 Symmetry of matrix model of medium with orthogonal eigenpolarizations.

2.3.1 Conditions on matrices elements

Eigenpolarizations are those polarization states of light that do not change when passing through a medium. The amplitude and the overall phase of the beam of light with an eigenpolarization do, however, change. These changes are described by the corresponding eigenvalues. In optics and electrodynamics the crystalline medium is characterized by the types of eigenpolarizations that this medium possesses. Because of that, ascertainment of generalized conditions for orthogonality of medium's eigenpolarizations allows determining the structure and symmetry of the matrix model for such class of media.

Using Eq.(3) we have

$$\chi_1\chi_2=-\frac{t_{12}}{t_{21}}. \tag{13}$$

Combining orthogonality condition in terms of the complex variables Eq.(4) with Eq.(3) we get first relation between non-diagonal elements of the Jones matrix in general case:

$$\boxed{|t_{12}| = |t_{21}|}. \tag{14}$$

Next, it can be seen that:

$$|\chi_1 + \chi_2|^2 = |\chi_1|^2 + |\chi_2|^2 + \chi_1\chi_2^* + \chi_2\chi_1^* = |\chi_1|^2 + \frac{1}{|\chi_1|^2} - 2 = \frac{\left(|\chi_1|^2 - 1\right)^2}{|\chi_1|^2} = \left(|\chi_1| - \frac{1}{|\chi_1|}\right)^2 = \left(|\chi_1| - |\chi_2|\right)^2 . \tag{15}$$

If we rewrite the relation for eigenvectors as:

$$\chi_{1,2} = \frac{e_1 \pm e_2}{2t_{21}}, \text{ where } e_1 = t_{22} - t_{11};\ e_2 = \sqrt{(t_{22} - t_{11})^2 + 4t_{12}t_{21}}, \tag{16}$$

then Eq.(15) takes the form:

$$\left|\frac{e_1}{t_{21}}\right| = \left|\frac{e_1 + e_2}{2t_{21}}\right| - \left|\frac{e_1 - e_2}{2t_{21}}\right|, \tag{17.a}$$

or

$$\frac{|e_1 + e_2| - |e_1 - e_2|}{|e_1|} = 2. \tag{17.b}$$

Let us present the parameters e_1 and e_2 in the exponential form:

$$e_{1,2} = \rho_{1,2} \exp(-i\psi_{1,2}). \tag{18}$$

Then, after some mathematics, from Eq.(17.b) we obtain:

$$\cos(\psi_1 - \psi_2)^2 = 1, \tag{19.a}$$

$$\psi_1 = \psi_2 \pm n\pi . \tag{19.b}$$

Combining the equations Eq.(16) and Eq.(19.b), for the case $n = 0$ we can write:

$$|e_2|^2 \exp(2i\psi_1) = |t_{22} - t_{11}|^2 \exp(2i\psi_1) + 4|t_{12}||t_{12}| \exp[i(\phi_{12} + \phi_{21})]. \tag{20}$$

From Eq.(20) it can be deduced that $2\psi_1 = \phi_{12} + \phi_{21} \pm n\pi$ or, taking into account Eq.(16), it is transformed into

$$\boxed{2\phi_{22-11} = \phi_{12} + \phi_{21} \pm n\pi}. \tag{21}$$

For sum of matrix elements t_{21} and t_{12}, allowing for Eq.(21), we can write:

$$t_{21}+t_{12}=\left|t_{21}+t_{12}\right|\exp(i\phi_{12+21})=\left|t_{12}\right|\left[\left(\cos\phi_{12}+\cos\phi_{21}\right)+i\left(\sin\phi_{12}+\sin\phi_{21}\right)\right]= \\ =2\left|t_{12}\right|\cos\frac{\phi_{12}-\phi_{21}}{2}\exp\left(i\frac{\phi_{12}+\phi_{21}}{2}\right), \quad (22)$$

from which it follows:

$$\boxed{2\phi_{21+12}=\phi_{12}+\phi_{21}\pm n\pi}, \quad (23)$$

excluding the case when $t_{21} = - t_{12}$.

Taking into account Eq.(14), one more relation on phases of difference of matrix elements t_{21} and t_{12} can be also written:

$$i\left(t_{21}-t_{12}\right)=\left|i\left(t_{21}-t_{12}\right)\right|\exp(i\phi_{21-12})=i\left|t_{12}\right|\left[\left(\cos\phi_{12}-\cos\phi_{21}\right)+i\left(\sin\phi_{12}-\sin\phi_{21}\right)\right]= \\ =2\left|t_{12}\right|\sin\frac{\phi_{21}-\phi_{12}}{2}\exp\left(i\frac{\phi_{12}+\phi_{21}}{2}\right) \quad (24)$$

From Eq.(24) it follows:

$$\boxed{2\phi_{i(21-12)}=\phi_{12}+\phi_{21}\pm n\pi}. \quad (25)$$

Using interrelation Eq.(8) between Jones and Mueller methods and description of spectral problem as Eq.(9) we can also study symmetry of the Mueller matrix of medium with orthogonal polarizations.

In particular, from Eq.(8) it follows:

$$\left|t_{21}\right|=\sqrt{\frac{m_{11}-m_{21}+m_{12}-m_{22}}{2}};\quad \left|t_{12}\right|=\sqrt{\frac{m_{11}+m_{21}-m_{12}-m_{22}}{2}}; \\ \left|t_{11}\right|=\sqrt{\frac{m_{11}+m_{21}+m_{12}+m_{22}}{2}};\quad \left|t_{22}\right|=\sqrt{\frac{m_{11}-m_{21}-m_{12}+m_{22}}{2}}; \quad (26)$$

$$\cos(\phi_{12})=\frac{m_{13}+m_{23}}{2\left|t_{11}\right|\left|t_{12}\right|};\quad \sin(\phi_{12})=\frac{m_{14}+m_{24}}{2\left|t_{11}\right|\left|t_{12}\right|}; \\ \cos(\phi_{21})=\frac{m_{31}+m_{32}}{2\left|t_{11}\right|\left|t_{21}\right|};\quad \sin(\phi_{21})=\frac{m_{41}+m_{42}}{2\left|t_{11}\right|\left|t_{21}\right|}; \\ \cos(\phi_{22})=\frac{m_{33}+m_{44}}{2\left|t_{11}\right|\left|t_{22}\right|};\quad \sin(\phi_{22})=\frac{m_{43}-m_{34}}{2\left|t_{11}\right|\left|t_{12}\right|}. \quad (27)$$

Here we assumed that all phases of Jones matrix elements Eq.2 are normalized on phase of the first matrix element (i.e. $\phi_{mn}\to\phi_{mn}-\phi_{11}$). Thus, in this case the phase of element m_{11} is $\phi_{11}\to 0$.

Condition Eq.(14) for orthogonality of eigenpolarizations of medium in terms of Jones matrix elements can be transformed using Eq.(24) into relation:

$$m_{11}-m_{21}+m_{12}-m_{22}=m_{11}+m_{21}-m_{12}-m_{22},$$

or

$$\boxed{m_{12} = m_{21}}. \tag{28}$$

Condition Eq.(21) in terms of Jones matrix elements can be transformed using Eq.(27) into relations:

$$t_{mn} = |t_{mn}|(\cos(\phi_{mn}) + i\sin(\phi_{mn}));$$

$$\frac{\sin(\phi_{12}+\phi_{21})}{\cos(\phi_{12}+\phi_{21})} = \mathrm{tg}\left(2\,\mathrm{arctg}\left(\frac{\mathrm{Im}(t_{22}-t_{11})}{\mathrm{Re}(t_{22}-t_{11})}\right)\right);$$

$$\mathrm{tg}(\phi_{22-11}) = \frac{\mathrm{Im}(t_{22}-t_{11})}{\mathrm{Re}(t_{22}-t_{11})} = \frac{m_{43}-m_{34}}{m_{33}+m_{44}-m_{11}-m_{21}-m_{12}-m_{22}};$$

$$\cos(\phi_{12}+\phi_{21}) = \cos(\phi_{12})\cos(\phi_{21}) - \sin(\phi_{12})\sin(\phi_{21});$$

$$\sin(\phi_{12}+\phi_{21}) = \cos(\phi_{12})\sin(\phi_{21}) + \cos(\phi_{21})\sin(\phi_{12});$$

$$\cos(\phi_{12}+\phi_{21}) = \frac{m_{13}+m_{23}}{2|t_{11}||t_{12}|}\frac{m_{31}+m_{32}}{2|t_{11}||t_{21}|} + \frac{m_{14}+m_{24}}{2|t_{11}||t_{12}|}\frac{m_{41}+m_{42}}{2|t_{11}||t_{21}|};$$

$$\sin(\phi_{12}+\phi_{21}) = \frac{m_{13}+m_{23}}{2|t_{11}||t_{12}|}\frac{m_{41}+m_{42}}{2|t_{11}||t_{21}|} - \frac{m_{14}+m_{24}}{2|t_{11}||t_{12}|}\frac{m_{31}+m_{32}}{2|t_{11}||t_{21}|},$$

or in terms of Mueller matrix elements

$$\boxed{\begin{aligned}&\frac{(m_{41}+m_{42})(m_{23}+m_{13})-(m_{14}+m_{24})(m_{32}+m_{31})}{(m_{41}+m_{42})(m_{14}+m_{24})+(m_{23}+m_{13})(m_{32}+m_{31})} = \\ &\qquad = \mathrm{tg}\left[2\,\mathrm{arctg}\left(\frac{m_{43}-m_{34}}{m_{11}+m_{21}+m_{12}+m_{22}-m_{33}-m_{44}}\right)\right]\end{aligned}} \tag{29}$$

Expression Eq.(29) is evidently inconvenient for practical use. Thus, we propose to use relations $2\phi_{11+22} = 2\phi_{(21+12)} = 2\phi_{i(21+12)} = \phi_{12} + \phi_{21} \pm n\pi$ (see Eq.(21), Eq.(23), Eq.(25)). Then:

$$\frac{\mathrm{Im}(i(t_{21}-t_{12}))}{\mathrm{Re}(i(t_{21}-t_{12}))} = \frac{\mathrm{Im}(t_{21}+t_{12})}{\mathrm{Re}(t_{21}+t_{12})};$$

$$i(t_{21}-t_{12}) = |t_{12}|\left[i\left(\frac{m_{31}+m_{32}}{2|t_{11}||t_{12}|} - \frac{m_{13}+m_{23}}{2|t_{11}||t_{12}|}\right) - \frac{m_{41}+m_{42}}{2|t_{11}||t_{12}|} - \frac{m_{14}+m_{24}}{2|t_{11}||t_{12}|}\right] =$$

$$= \frac{1}{2|t_{11}|}[i(m_{31}+m_{32}-m_{13}-m_{23}) - m_{41}-m_{42}-m_{14}-m_{24}];$$

$$t_{21}+t_{12} = |t_{12}|\left[\frac{m_{13}+m_{23}}{2|t_{11}||t_{12}|} + \frac{m_{31}+m_{32}}{2|t_{11}||t_{12}|} + i\left(-\frac{m_{14}+m_{24}}{2|t_{11}||t_{12}|} + \frac{m_{41}+m_{42}}{2|t_{11}||t_{12}|}\right)\right] =$$

$$= \frac{1}{2|t_{11}|}[m_{31}+m_{32}+m_{13}+m_{23} + i(m_{41}+m_{42}-m_{14}-m_{24})];$$

$$\frac{\mathrm{Im}(i(t_{21}-t_{12}))}{\mathrm{Re}(i(t_{21}-t_{12}))} = \frac{\mathrm{Im}(t_{21}+t_{12})}{\mathrm{Re}(t_{21}+t_{12})} \Rightarrow$$

$$\boxed{\frac{m_{13}+m_{23}-m_{31}-m_{32}}{m_{41}+m_{42}+m_{14}+m_{24}}=\frac{m_{41}+m_{42}-m_{14}-m_{24}}{m_{13}+m_{23}+m_{31}+m_{32}}}, \tag{30}$$

or

$$\boxed{(m_{13}+m_{23})^2-(m_{31}+m_{32})^2=(m_{41}+m_{42})^2-(m_{14}-m_{24})^2} \tag{30'}$$

Finally we have

$$\boxed{\frac{m_{13}+m_{23}-m_{31}-m_{32}}{m_{41}+m_{42}+m_{14}+m_{24}}=\frac{m_{41}+m_{42}-m_{14}-m_{24}}{m_{13}+m_{23}+m_{31}+m_{32}}=\frac{m_{43}-m_{34}}{m_{11}+m_{21}+m_{12}+m_{22}-m_{33}-m_{44}}} \tag{31}$$

Spectral problem in scope of Mueller formalism can be expressed in form Eq.(9) where the fact that eigenpolarizations **S** of the Mueller matrix are orthogonal was already taken into account. In other words Eq.(9) holds for following normalized Stokes vectors :

$$\mathbf{S}_1=\begin{pmatrix}1\\ s_2\\ s_3\\ s_4\end{pmatrix};\quad \mathbf{S}_2=\begin{pmatrix}1\\ -s_2\\ -s_3\\ -s_4\end{pmatrix}. \tag{32}$$

For Stokes vectors Eq.(32) one can write:

$$\begin{pmatrix}m_{11}&m_{12}&m_{13}&m_{14}\\ m_{21}&m_{22}&m_{23}&m_{24}\\ m_{31}&m_{32}&m_{33}&m_{34}\\ m_{41}&m_{42}&m_{43}&m_{44}\end{pmatrix}\begin{pmatrix}1\\ s_2\\ s_3\\ s_4\end{pmatrix}=\begin{pmatrix}m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4\\ m_{21}+m_{22}s_2+m_{23}s_3+m_{24}s_4\\ m_{31}+m_{32}s_2+m_{33}s_3+m_{34}s_4\\ m_{41}+m_{42}s_2+m_{43}s_3+m_{44}s_4\end{pmatrix}=\underbrace{(m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4)}_{\nu_1}\begin{pmatrix}1\\ s_2\\ s_3\\ s_4\end{pmatrix};$$

$$\begin{pmatrix}m_{11}&m_{12}&m_{13}&m_{14}\\ m_{21}&m_{22}&m_{23}&m_{24}\\ m_{31}&m_{32}&m_{33}&m_{34}\\ m_{41}&m_{42}&m_{43}&m_{44}\end{pmatrix}\begin{pmatrix}1\\ -s_2\\ -s_3\\ -s_4\end{pmatrix}=\begin{pmatrix}m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4\\ m_{21}-m_{22}s_2-m_{23}s_3-m_{24}s_4\\ m_{31}-m_{32}s_2-m_{33}s_3-m_{34}s_4\\ m_{41}-m_{42}s_2-m_{43}s_3-m_{44}s_4\end{pmatrix}=\underbrace{(m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4)}_{\nu_2}\begin{pmatrix}1\\ -s_2\\ -s_3\\ -s_4\end{pmatrix}; \tag{33}$$

From Eq.(33) it follows:

$$\begin{pmatrix}m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4\\ m_{21}+m_{22}s_2+m_{23}s_3+m_{24}s_4\\ m_{31}+m_{32}s_2+m_{33}s_3+m_{34}s_4\\ m_{41}+m_{42}s_2+m_{43}s_3+m_{44}s_4\end{pmatrix}-\underbrace{(m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4)}_{\nu_1}\begin{pmatrix}1\\ s_2\\ s_3\\ s_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}; \tag{34}$$

$$\begin{pmatrix}m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4\\ m_{21}-m_{22}s_2-m_{23}s_3-m_{24}s_4\\ m_{31}-m_{32}s_2-m_{33}s_3-m_{34}s_4\\ m_{41}-m_{42}s_2-m_{43}s_3-m_{44}s_4\end{pmatrix}-\underbrace{(m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4)}_{\nu_2}\begin{pmatrix}1\\ -s_2\\ -s_3\\ -s_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}; \tag{35}$$

or

$$
\begin{aligned}
&m_{21}+m_{22}s_2+m_{23}s_3+m_{24}s_4-s_2(m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4)=0\\
&m_{31}+m_{32}s_2+m_{33}s_3+m_{34}s_4-s_3(m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4)=0\\
&m_{41}+m_{42}s_2+m_{43}s_3+m_{44}s_4-s_4(m_{11}+m_{12}s_2+m_{13}s_3+m_{14}s_4)=0\\
&m_{21}-m_{22}s_2-m_{23}s_3-m_{24}s_4-s_2(m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4)=0\\
&m_{31}-m_{32}s_2-m_{33}s_3-m_{34}s_4-s_3(m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4)=0\\
&m_{41}-m_{42}s_2-m_{43}s_3-m_{44}s_4-s_4(m_{11}-m_{12}s_2-m_{13}s_3-m_{14}s_4)=0
\end{aligned}
$$

By adding Eq.(34) to Eq.(35) we obtain:

$$
\begin{aligned}
m_{21}&=s_2(m_{12}s_2+m_{13}s_3+m_{14}s_4),\\
m_{31}&=s_3(m_{12}s_2+m_{13}s_3+m_{14}s_4),\\
m_{41}&=s_4(m_{12}s_2+m_{13}s_3+m_{14}s_4).
\end{aligned}
\tag{36}
$$

Multiplying Eq.(36) by s_2, s_3, s_4, respectively, and, taking into account that for completely polarized radiation normalized Stokes parameters obey the condition $\left(s_2{}^2+s_3{}^2+s_4{}^2\right)^{1/2}=1$, we have:

$$
s_2\left(m_{21}-m_{12}\right)+s_3\left(m_{31}-m_{13}\right)+s_4\left(m_{41}-m_{14}\right)=0\,,\tag{37}
$$

or

$$
m_{21}\left(m_{21}-m_{12}\right)+m_{31}\left(m_{31}-m_{13}\right)+m_{41}\left(m_{41}-m_{14}\right)=0\,,\tag{38}
$$

$$
\frac{m_{12}{}^2+m_{13}{}^2+m_{14}{}^2}{m_{12}m_{21}+m_{13}m_{31}+m_{14}m_{41}}=1\,.\tag{39}
$$

Taking into account that for Mueller-Jones matrix the following condition

$$
m_{12}{}^2+m_{13}{}^2+m_{14}{}^2=m_{21}{}^2+m_{31}{}^2+m_{41}{}^2\,,\tag{40}
$$

is true [34], from Eq.(39) and Eq.(40) we can write that:

$$
\begin{gathered}
m_{12}{}^2+m_{13}{}^2+m_{14}{}^2+m_{21}{}^2+m_{31}{}^2+m_{41}{}^2=2(m_{12}m_{21}+m_{13}m_{31}+m_{14}m_{41})\Rightarrow\\
\Rightarrow\left(m_{12}-m_{21}\right)^2+\left(m_{13}-m_{31}\right)^2+(m_{14}-m_{41})^2=0
\end{gathered}
\tag{41}
$$

from Eq.(41) it results that:

$$
\boxed{\begin{aligned}
m_{12}&=m_{21},\\
m_{13}&=m_{31},\\
m_{14}&=m_{41}.
\end{aligned}}
\tag{42}
$$

From Eq.(42) it follows that for complete description of anisotropy of medium with orthogonal eigenpolarizations the knowledge of second, third and fourth columns(rows) of Mueller matrix is sufficient. This result can be used during optimization of polarimeter for studying of given polarization class of media.

2.3.2 Orthogonality conditions in terms of anisotropy parameters.

To derive orthogonality condition in terms of anisotropy parameters we use the mentioned matrix model of arbitrary homogeneous anisotropy (deterministic) Eq.(10), that has recently been presented in [33].

The relations Eqs.(16), Eq.(22) and Eq.(24) appear useful to rewrite in the form:

$$t_{22} - t_{11} = (a_1 + ib_1)\exp\left(-i\frac{\Delta}{2}\right), \tag{43}$$

$$t_{21} - t_{12} = (a_2 + ib_2)\exp\left(-i\frac{\Delta}{2}\right), \tag{44}$$

$$t_{21} + t_{12} = (a_3 + ib_3)\exp\left(-i\frac{\Delta}{2}\right), \tag{45}$$

Where, in compliance with Eqs.(10) and Eq.(11), parameters a_i and b_i have the following form:

$$\begin{aligned}
a_1 &= (1+P)R\sin\frac{\Delta}{2}\sin(2\alpha-\varphi) - (1-P)\cos\frac{\Delta}{2}\cos(2\gamma-\varphi),\\
b_1 &= (1-P)R\cos\frac{\Delta}{2}\sin(2\gamma-\varphi) - (1+P)\sin\frac{\Delta}{2}\cos(2\alpha-\varphi),\\
a_2 &= (1-P)R\sin\frac{\Delta}{2}\cos(2\alpha-2\gamma-\varphi) - (1+P)\cos\frac{\Delta}{2}\sin(\varphi),\\
b_2 &= (1+P)R\cos\frac{\Delta}{2}\cos(\varphi) - (1-P)\sin\frac{\Delta}{2}\sin(2\alpha-2\gamma-\varphi),\\
a_3 &= (1+P)R\sin\frac{\Delta}{2}\cos(2\alpha-\varphi) + (1-P)\cos\frac{\Delta}{2}\sin(2\gamma-\varphi),\\
b_3 &= (1-P)R\cos\frac{\Delta}{2}\cos(2\gamma-\varphi) + (1+P)\sin\frac{\Delta}{2}\sin(2\alpha-\varphi).
\end{aligned} \tag{46}$$

Combining Eqs.(44) and (45), equation (14) can be written:

$$a_2 a_3 + b_2 b_3 = 0. \tag{47}$$

As it follows from Eq.(21), Eq.(23) and Eq.(25)

$$\phi_{22-11} = \phi_{12+21} = \phi_{i(21-12)}, \tag{48}$$

Combining Eq.(43), Eq.(45) and Eq.(48) and further Eq.(43), Eq.(44) and Eq.(48) we obtain:

$$a_1 b_3 - a_3 b_1 = 0, \tag{49}$$

$$a_1 b_2 + a_2 b_1 = 0. \tag{50}$$

Note that relation Eq.(47) can be obtained by combining Eq.(44), Eq.(45) and Eq.(48) as well.

For further practical usefulness the relations Eq.(47), Eq.(49), Eq.(50) can be rewritten in the form:

$$
\begin{aligned}
&a_1 b_3 - a_3 b_1 = 0\,,\\
&(a_1 b_2 + a_2 b_1)\sin 2(\alpha - \varphi) + (a_2 a_3 + b_2 b_3)\cos 2(\alpha - \varphi) = 0\,,\\
&(a_1 b_2 + a_2 b_1)\cos 2(\alpha - \varphi) - (a_2 a_3 + b_2 b_3)\sin 2(\alpha - \varphi) = 0\,.
\end{aligned}
\tag{51}
$$

Finally, substitution Eq.(46) in Eq.(51) gives the relations for determining the values of anisotropy parameters for medium having the orthogonal eigenpolarizations:

$$
(1-P)^2 R\cos^2\frac{\Delta}{2} - (1-P^2)(1-R^2)\sin\frac{\Delta}{2}\cos\frac{\Delta}{2}\sin 2(\alpha-\gamma) - (1+P)^2 R\sin^2\frac{\Delta}{2} = 0\,, \tag{52}
$$

$$
(1-P)\left[(1+R^2)\cos 2(\alpha-\gamma-\phi) - (1-R^2)\cos 2(\alpha-\gamma)\right] = 0\,, \tag{53}
$$

$$
\begin{aligned}
&2\left(1+P^2\right)R\sin\frac{\Delta}{2}\cos\frac{\Delta}{2} + \\
&+\frac{1}{2}(1-P^2)\sin^2\frac{\Delta}{2}\left[(1-R^2)\sin 2(\alpha-\gamma) + (1+R^2)\sin 2(\alpha-\gamma-\varphi)\right] - \\
&-\frac{1}{2}(1-P^2)\cos^2\frac{\Delta}{2}\left[(1-R^2)\sin 2(\alpha-\gamma) - (1+R^2)\sin 2(\alpha-\gamma-\varphi)\right] = 0
\end{aligned}
\quad . \tag{54}
$$

From Eq.(52) it is follows=>

$$
\boxed{\Delta = -2\,\mathrm{arctg}\left(\frac{1}{2R}\frac{P-1}{P+1}\left[(1-R^2)\sin 2(\alpha-\gamma) - \sqrt{\sin^2 2(\alpha-\gamma)\left(1-R^2\right)^2 + 4R^2}\right]\right)} \tag{55}
$$

$$
\boxed{\varphi = \alpha - \gamma + \frac{1}{2}\arccos\left[\cos 2(\alpha-\gamma)\frac{R^2-1}{R^2+1}\right] - \frac{\pi}{2}} \tag{56}
$$

Eq.(52) and Eq.(54) can be represented as

$$
\begin{aligned}
&-2PR + \left(1+P^2\right)R\cos\Delta + \frac{1}{2}\left(1-P^2\right)\left(1-R^2\right)\sin\Delta\sin 2(\alpha-\gamma) = 0\,,\\
&\left(1+P^2\right)R\sin\Delta + \frac{1}{2}\left(1-P^2\right)\left[\left(1+R^2\right)\sin 2(\alpha-\gamma-\varphi) - \left(1-R^2\right)\sin 2(\alpha-\gamma)\cos\Delta\right] = 0\,.
\end{aligned}
\tag{57}
$$

Making the following designations

$$
\begin{aligned}
&A = \frac{1}{2}\left(1-P^2\right)\left(1-R^2\right)\sin 2(\alpha-\gamma); \qquad B = \left(1+P^2\right)R;\\
&C = -2PR; \qquad D = -\frac{1}{2}\left(1-P^2\right)\left(1+R^2\right)\sin 2(\alpha-\gamma-\varphi),
\end{aligned}
\tag{58}
$$

then Eqs.(57) take the forms

$$A\sin\Delta + B\cos\Delta + C = 0,$$
$$A\cos\Delta - B\sin\Delta + D = 0. \tag{59}$$

For Eq.(59) to be satisfied the following equality is required

$$A^2 + B^2 = C^2 + D^2 . \tag{60}$$

Taking to account Eqs.(59) and (60) we have:

$$(C - B)\tan^2\frac{\Delta}{2} + 2A\tan\frac{\Delta}{2} + (C + B) = 0,$$
$$(D - A)\tan^2\frac{\Delta}{2} - 2B\tan\frac{\Delta}{2} + (D + A) = 0. \tag{61}$$

Solution of the system Eqs.(61) is

$$\operatorname{tg}\frac{\Delta}{2}_{1,2} = \frac{-A \pm \sqrt{A^2 - C^2 + B^2}}{C - B} = \frac{-A \pm \sqrt{D^2 - B^2 + B^2}}{C - B} = \begin{cases} \dfrac{-A + D}{C - B} \\ -\dfrac{A - D}{C - B} \end{cases}, \tag{62}$$

$$\operatorname{tg}\frac{\Delta}{2}_{1,2} = \frac{B \pm \sqrt{B^2 - D^2 + A^2}}{D - A} = \frac{B \pm \sqrt{B^2 - B^2 + C^2}}{D - A} = \begin{cases} \dfrac{B + C}{D - A} \\ \dfrac{B - C}{D - A} \end{cases}. \tag{63}$$

From Eq.(60) it follows:

$$-\frac{A + D}{C - B} = \frac{B + C}{D - A},$$
$$-\frac{A + D}{C - B} \neq \frac{B - C}{D - A}. \tag{64}$$

Evidently, we need to choose in Eqs.(62) and (63) the same solutions, then

$$\operatorname{tg}\frac{\Delta}{2} = \frac{A + D}{B - C} = \operatorname{tg}\frac{\Delta}{2} = \frac{-A + \sqrt{A^2 - C^2 + B^2}}{C - B}. \tag{65}$$

As it can be seen Eq.(65) is equal to Eq.(55). Thus, we have 2 equations and 6 unknown parameters to be determined. Therefore, 4 out of 6 parameters we should fix and, then, other 2 can be determined from Eq.(52) and Eq.(53). It turns convenient to determine the parameters Δ and φ (see Eqs. (55) and (56)), while other 4 parameters are fixed in a range of definition.

For depolarizing Mueller matrix interpretation the method of coherency matrix introduced by Cloude [35,36] has widely been used. According to this method an arbitrary Mueller matrix **M** including depolarization and errors effect are represented as a sum of four nondepolarizing (deterministic) Muller matrices $\mathbf{M}_D^{1-4}$:

$$\mathbf{M} = \sum_{k=1}^{4} \mu_k \mathbf{M}_D^k . \tag{66}$$

In Eq.(66) μ_k – are eigenvalues of the Cloude's coherency matrix playing the role of weighting multipliers. So, in compliance with Eq.(66) the anisotropic properties of object are presented by simultaneous parallel independent effects of four deterministic anisotropic parts.

If in Eq.(66) only one eigenvalue μ_k is non-zero, the corresponding Mueller matrix

represents nondepolarizing (homogeneous anisotropic) medium. If all four eigenvalues of coherence matrix are non-zero the dominant type of deterministic effect of object on input polarization is described by Mueller matrix associated with the maximum eigenvalue in sum Eq.(66). The case of negative eigenvalues indicates that the Muller matrix describes a physically impossible transformation. This may occur as a result of measurement errors. If eigenvalues of coherency matrix are comparable, then the studied medium depolarizes input radiation completely. In this case dominant type of deterministic behaviour of studied medium can not be determined.

Our previous studies have shown that direct using of the generalized equivalence theorem for the analysis of "raw" experimental Mueller matrices can lead to significant errors in the results analysis. This means that before interpreting initial experimental Mueller matrix should be filtered out to exclude negative eigenvalues of coherency matrix.

The elements of Cloude's coherency and Mueller matrix are related as follows [35,36]:

$$\begin{aligned}
c_{11} &= \frac{1}{4}\left(m_{11}+m_{22}+m_{33}+m_{44}\right), \quad c_{21} = \frac{1}{4}\left(m_{12}+m_{21}+im_{34}-im_{43}\right), \\
c_{12} &= \frac{1}{4}\left(m_{12}+m_{21}-im_{34}+im_{43}\right), \quad c_{22} = \frac{1}{4}\left(m_{11}+m_{22}-m_{33}-m_{44}\right), \\
c_{13} &= \frac{1}{4}\left(m_{13}+m_{31}+im_{24}-im_{42}\right), \quad c_{23} = \frac{1}{4}\left(im_{14}+m_{23}+m_{32}-im_{41}\right), \\
c_{14} &= \frac{1}{4}\left(m_{14}-im_{23}+im_{32}+m_{41}\right), \quad c_{24} = \frac{1}{4}\left(-im_{13}+im_{31}+m_{24}+m_{42}\right), \\
c_{31} &= \frac{1}{4}\left(m_{13}+m_{31}-im_{24}+im_{42}\right), \quad c_{41} = \frac{1}{4}\left(m_{14}+im_{23}-im_{32}+m_{41}\right), \\
c_{32} &= \frac{1}{4}\left(-im_{14}+m_{23}+m_{32}+im_{41}\right), \quad c_{42} = \frac{1}{4}\left(im_{13}-im_{31}+m_{24}+m_{42}\right), \\
c_{33} &= \frac{1}{4}\left(m_{11}-m_{22}+m_{33}-m_{44}\right), \quad c_{43} = \frac{1}{4}\left(-im_{12}+im_{21}+m_{34}+m_{43}\right), \\
c_{34} &= \frac{1}{4}\left(im_{12}-im_{21}+m_{34}+m_{43}\right), \quad c_{44} = \frac{1}{4}\left(m_{11}-m_{22}-m_{33}+m_{44}\right).
\end{aligned} \tag{67}$$

Coherency matrix $\mathbf{C}$ has always four real eigenvalues, since it is Hermitian, i.e., $c_{i,j} = c_{i,j}^{*}$, where $*$ denotes Hermitian conjugation. Four eigenvectors $\mathbf{\Psi}^{1-4}$ of the matrix $\mathbf{C}$ are directly related to the Jones matrices $\mathbf{T}^{1-4}$ as follow:

$$\left.\begin{aligned}
t_{11}^{(k)} &= \Psi_1^{(k)} + \Psi_2^{(k)}, \quad t_{12}^{(k)} = \Psi_3^{(k)} - i\Psi_4^{(k)}, \\
t_{21}^{(k)} &= \Psi_3^{(k)} + i\Psi_4^{(k)}, \quad t_{22}^{(k)} = \Psi_1^{(k)} - \Psi_2^{(k)};
\end{aligned}\right| \quad k = 1 \div 4; \tag{68}$$

where k - number of eigenvectors of coherency matrix $\mathbf{C}$.

The initial depolarizing Mueller matrix is represented by the sum of four deterministic Muller matrices $\mathbf{M}_D^{1-4}$ corresponding to the Jones matrices Eq.(68).

Eigenvalues of the coherency matrix $\mathbf{C}$ are used to characterise the depolarization properties of the studied object by single-value depolarization metric called entropy *H*.

$$H = -\sum_{r=1}^{N} K_r \log_N (K_r); \;\; K_r = \frac{\mu_r}{\sum_{j=1}^{4} \mu_j}, \tag{69}$$

where $N = \overline{1,4}$ is chosen such that the condition $0 \le H \le 1$ is satisfied. In the case when all eigenvalues of the coherency matrix are positive one should set $N = 4$. In other cases, N is the number of positive eigenvalues of the coherency matrix **C**.

Entropy H defined in Eq.(69) characterizes the degree of anisotropic disorder yielding depolarization of incident radiation. If $H = 0$, i.e., only one eigenvalue of coherence matrix is nonzero, the studied medium described by corresponding deterministic Mueller matrix is homogeneous and doesn't depolarize input radiation. If $H = 1$, the studied medium is ideal depolarizer. Intermediate values of H correspond to anisotropic depolarizing medium that in addition to the depolarization, change the polarization of input radiation. As it was noted above, the author [35,36] suggested that a medium leading to depolarization can be characterized by the some dominant effective anisotropy in case of $H < 0.5$. If in Eq.(66) all four components have comparable weighting multipliers, all of them contribute equally to the change of polarization of radiation.

2.4 Development of optimized polarimeter

In given section we describe an assembled polarimeter that is optimized for accurate measurement of all elements of Mueller matrix of media. As we noted early it would be extremely desirable that this polarimeter could dynamically adjust to exact experimental conditions and have no moving units. This will provide a minimal measurement time with maximum accuracy and, hence, be a promising basis for new type of imaging Mueller polarimetry. One of possible ways to realize dynamically adjusted polarimeter which we implemented here is using

LC retarders as polarization transducers, which retardation is electrically variable - Liquid Crystal Variable Retarders (LCVR). As it was shown in [16] a minimal number of LC transducers to measure complete Mueller (all of sixteen matrix elements) matrix is four.

Mentioned LC transducers (Fig.1.) can replace an entire series of polymer and ordinary crystalline retarders. With new Swift LC Meadowlark technology (www.meadowlark.com), the switching speed is symmetric and approximately 150 microseconds.

Fig.1. Meadowlark's Liquid Crystal Variable Retarder (LCVR).

The Four Channel Digital Interface D3050 (Fig.2) designed for computer control of up to four Meadowlark Optics nematic liquid crystal devices at the same time. Package allows the amplitude of the 2 kHz square wave output to be driven either by an external DC analog signal supplied to a front panel connector or specific CellDRIVE (Fig.3) generated waveforms including sinusoidal, square, triangle, sawtooth and transient nematic effect waveforms. Additional functions include the capability to output a sync pulse on a front panel connector at desired points in the CellDRIVE generated waveforms and the ability to save/restore all CellDRIVE settings to/from a file.

Fig.2. Meadowlark's Four Channel Digital Interface.

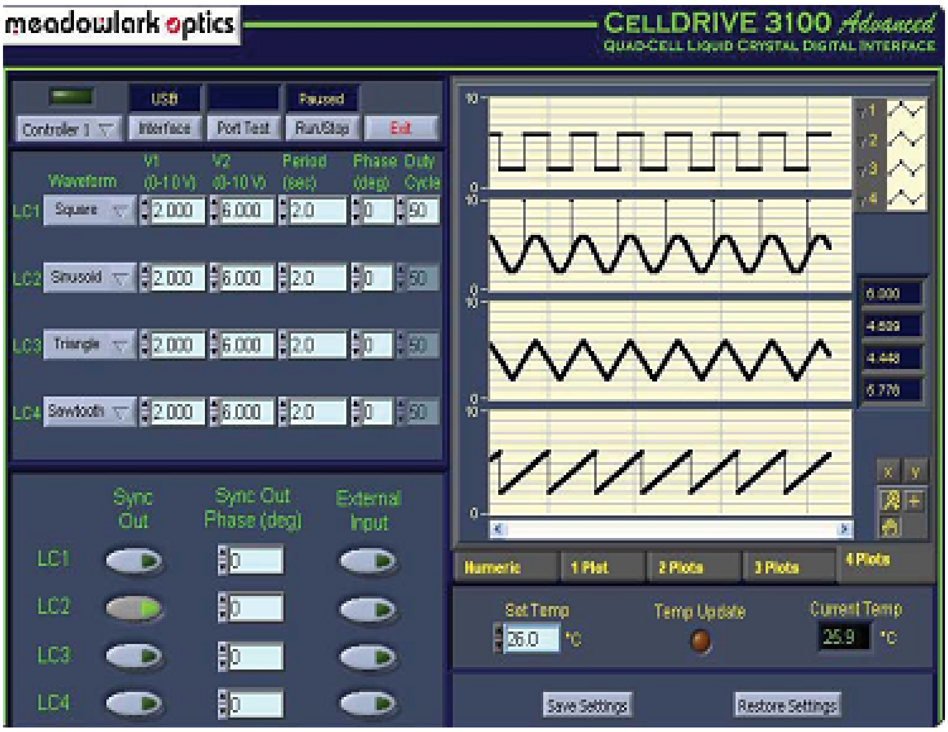


Fig.3. CellDRIVE 3100 advanced software

The tests of purchased LC transducers (for the results see Fig.4) was realized on automatized Mueller-polarimeter which operates with rotatable crystal plates at input and output channels (Fig.5).

Fig.4 shows nonzero anisotropies of LC transducers during applied voltage changing. It can be seen that besides main type of anisotropy (linear birefringece - Δ) the retarder's cell have also dependence of difference in transmittance for eigenpolarizations (linear dichroism) versus

applied voltage. This is very important fact which must to be taken in to account during polarimeters designing.

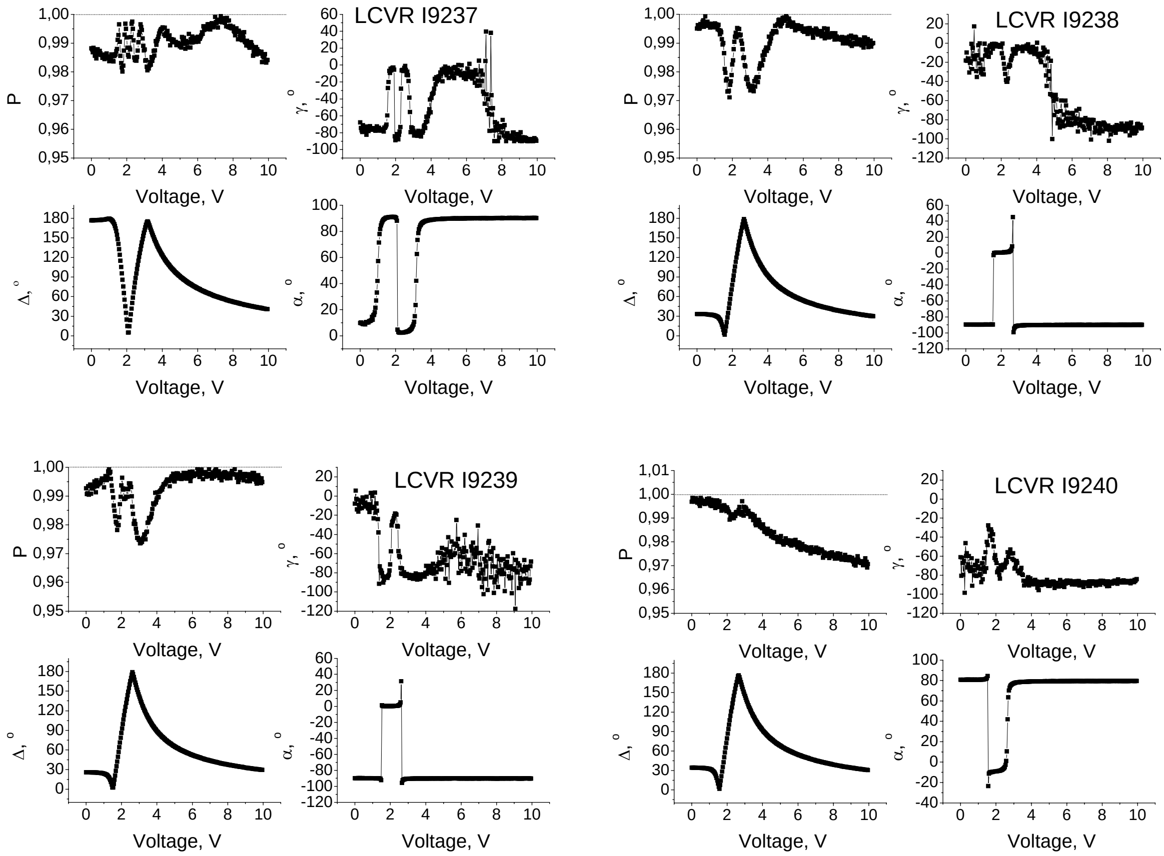


Fig.4. Dependences of LC transducers parameters on value of applied voltage. Analysis is performed in conformity with theorem Eq.(10).

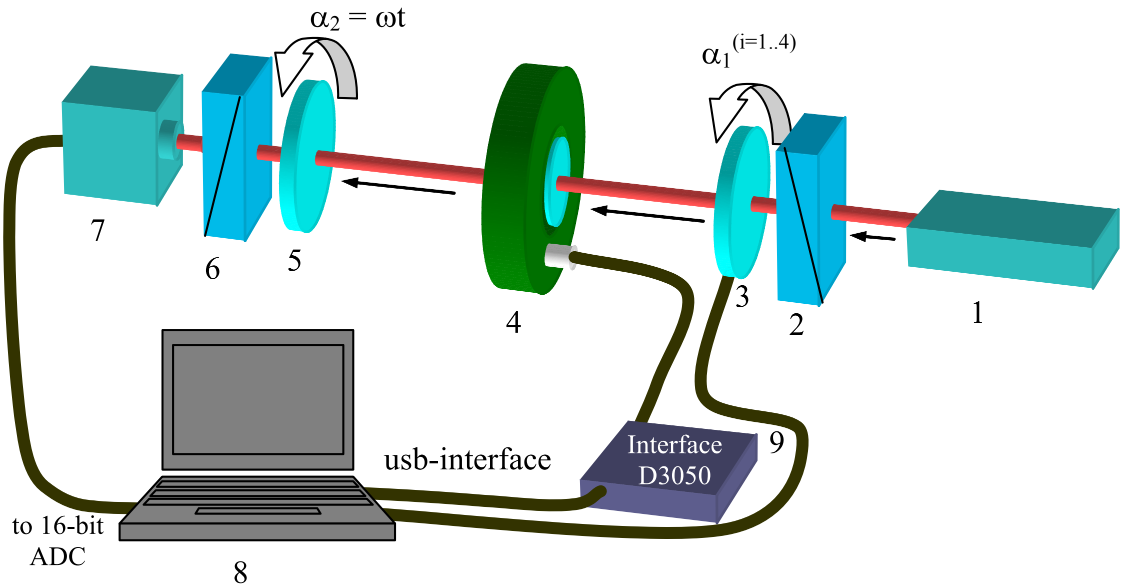


Fig.5. Scheme of Mueller matrix polarimeter used for LC transducers tests.

1. He-Ne laser;
2. polarizer;
3. rotatable crystal phase plate which orientation during measurement periodically consequently takes the discrete values (α_1^i);
4. LC transducers tested;

5. continuous rotatable crystal phase plate;
6. analyzer;
7. photodetector;
8. PC;
9. digital interface D3050.

Figure 6 shows the scheme of Mueller-polarimeter. Receiving channel (PSA) is a complete Stokes-polarimeter using two LC transducers (LCVR3 and LCVR4), described above. In the probing channel (PSG) consists of two LC transducers (LCVR1 and LCVR2) as well. Orientations of all units in polarimeter scheme are indicated in Fig.6. Mathematics of polarimeter operation is presented below.

$$\underbrace{\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} r_1^1 & r_1^2 & r_1^3 & r_1^4 \\ r_2^1 & r_2^2 & r_2^3 & r_2^4 \\ r_3^1 & r_3^2 & r_3^3 & r_3^4 \\ r_4^1 & r_4^2 & r_4^3 & r_4^4 \end{pmatrix}}_{\mathbf{G}} = \begin{pmatrix} s_1^1 & s_1^2 & s_1^3 & s_1^4 \\ s_2^1 & s_2^2 & s_2^3 & s_2^4 \\ s_3^1 & s_3^2 & s_3^3 & s_3^4 \\ s_4^1 & s_4^2 & s_4^3 & s_4^4 \end{pmatrix}, \quad (70)$$

where **M** is a Mueller matrix of object, **G** – characteristic matrix of Mueller-polarimeter, r_j^i, s_j^i are j-th Stokes parameters for i-th polarization of input and transmitted light respectively.

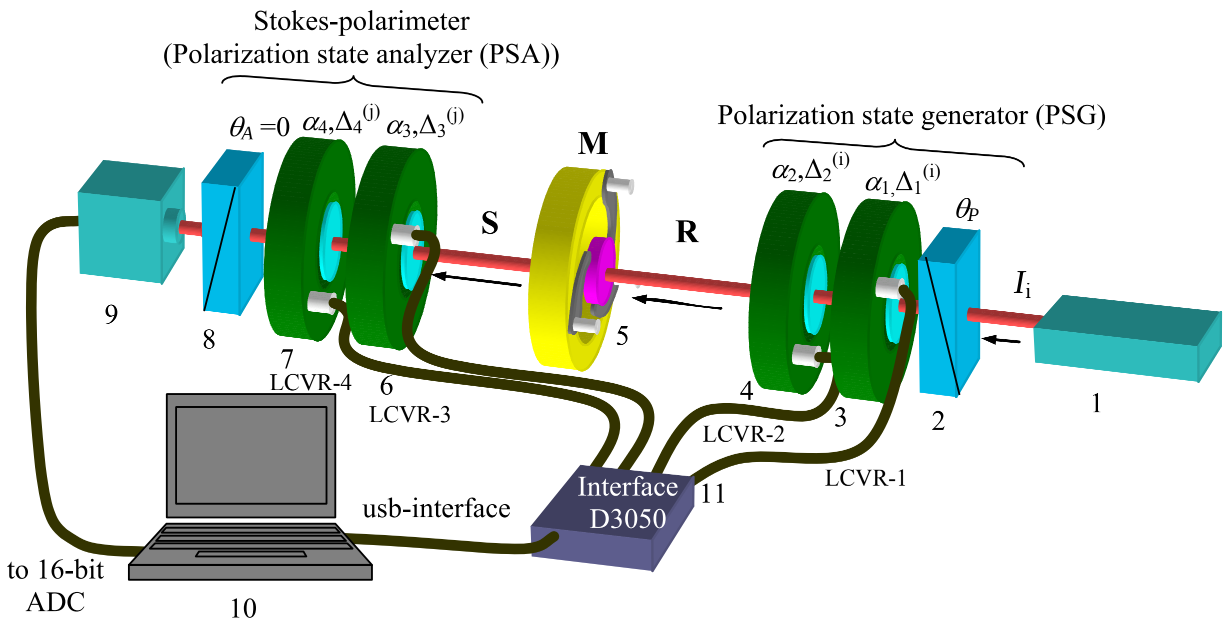


Fig.6. Mueller-polarimeter based on four LC transducers.
1. He-Ne laser;
2,8. polarizer (analyzer) with orientation θ_P (θ_A);
3,4,6,7. LC transducers (LCVR1-4) which orientations of fast axis's are $\alpha_{1\text{-}4}$ and introduced additional phase shifts are $\Delta_{1\text{-}4}$ (indexes i,j denotes changed phase shifts);
5. test sample;
9. photodetector;
10. PC;
11. digital interface D3050.
I_i – intensity registered by detector;

R, S – Stokes vectors of input and transmitted light correspondingly;
M – Mueller matrix of measured samples.

According to Eq.(70), measurement of the Mueller matrix by polarimeter Fig.6 is occurred by the means of sequential irradiation of sample by electromagnetic radiation with four different polarizations.

Through the fact that in Eq.(70) the Stokes parameters of output radiation are the measured parameters it needs to consider a PSA of polarimeter, Fig.6, in more details. PSA is a Stokes-polarimeter based on two LC transducers (see scheme Fig.7).

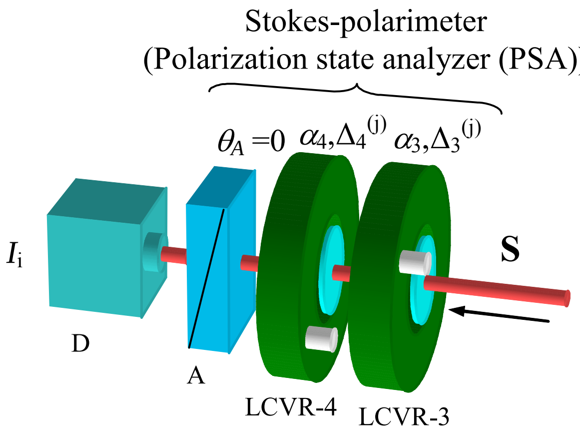


Fig.7. Stokes-polarimeter based on two LC transducers.
LCVR-3,4 – LC transducers;
A- analyzer;
D – detector;
S – Stokes vector of radiation to be analyzed.

Expression for intensity (i.e., first parameter of the Stokes vector) after analyzer A, Fig.7, can be written as

$$I = \left(\mathbf{M}_A(0°)\cdot\mathbf{M}_{\mathrm{LCVR}-4}(\Delta_4,\alpha_4)\cdot\mathbf{M}_{\mathrm{LCVR}-3}(\Delta_3,\alpha_3)\cdot\mathbf{S}\right)_1 . \tag{71}$$

where $\mathbf{M}_A(\theta)$, $\mathbf{M}_{\mathrm{LCVR}-3,4}(\Delta,\alpha)$ are the Mueller matrices of analyzer with orientation θ and LC transducers with phase shift δ and orientation α.

Due to the orientation α of LC transducer can not be changed electronically for measuring of all four Stokes parameters it is necessary to change the values of birefringence of LC transducers four times $\Delta_{1,2}^{(1-4)}$ at least. Then, we can set a system of equations relative to measured Stokes parameters:

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} \left(\mathbf{M}_A(0°)\cdot\mathbf{M}_{LCVR-4}(\Delta_4^{(1)},\alpha_4)\cdot\mathbf{M}_{LCVR-3}(\Delta_3^{(1)},\alpha_3)\cdot\mathbf{S}\right)_1 \\ \left(\mathbf{M}_A(0°)\cdot\mathbf{M}_{LCVR-4}(\Delta_4^{(2)},\alpha_4)\cdot\mathbf{M}_{LCVR-3}(\Delta_3^{(2)},\alpha_3)\cdot\mathbf{S}\right)_1 \\ \left(\mathbf{M}_A(0°)\cdot\mathbf{M}_{LCVR-4}(\Delta_4^{(3)},\alpha_4)\cdot\mathbf{M}_{LCVR-3}(\Delta_3^{(3)},\alpha_3)\cdot\mathbf{S}\right)_1 \\ \left(\mathbf{M}_A(0°)\cdot\mathbf{M}_{LCVR-4}(\Delta_4^{(4)},\alpha_4)\cdot\mathbf{M}_{LCVR-3}(\Delta_3^{(4)},\alpha_3)\cdot\mathbf{S}\right)_1 \end{bmatrix} \Rightarrow \mathbf{I} = \mathbf{B}(\Delta_3^{(i)},\alpha_3,\Delta_4^{(i)},\alpha_4)\cdot\mathbf{S} . \tag{72}$$

It is known [31] that the system Eq.(72) can be solved with minimal error when its condition number V_B is minimal:

$$V_B = \| \mathbf{B}(\Delta_3^{(i)},\alpha_3,\Delta_4^{(i)},\alpha_4) \| \; \| \mathbf{B}(\Delta_3^{(i)},\alpha_3,\Delta_4^{(i)},\alpha_4)^{-1} \| \to \min \Rightarrow \delta\mathbf{S} \to \delta\mathbf{S}_{\min} . \tag{73}$$

where || || denotes Euclidian (metric) norm and

$$\delta\mathbf{S} = \frac{\|\Delta\mathbf{S}\|}{\|\mathbf{S}_0\|}; \quad \mathbf{S} = \mathbf{S}_0 \pm \Delta\mathbf{S}. \tag{74}$$

In Eq.(74) $\mathbf{S}_0$ is exact (table) Stokes vector; $\Delta\mathbf{S}$ denotes vector of errors of corresponding Stokes parameters.

Eq.(73) results from the next relation [31]

$$\delta\mathbf{S} \le \frac{V_B \cdot (\delta\mathbf{B} + \delta\mathbf{I})}{1 - V_B \delta\mathbf{B}}; \tag{75}$$

where $\delta\mathbf{B}, \delta\mathbf{I}$ - are errors of determination of matrix $\mathbf{B}$ (called characteristic matrix of the system of linear equations or instrumental matrix of Stokes-polarimeter) and of vector $\mathbf{I}$ of intensities measured by detector D, see Fig.7.

We have found that the minimum value of condition number is $V_{\min} = 4.5$ and it is reached when angle between orientations of LC transducers is

$$|\alpha_4 - \alpha_3| = 22.5^0 \tag{76}$$

and its phase shifts are

$$\begin{aligned} \Delta_3^{(1-4)} &= (3.2°,\ 162.5°,\ 32.6°,\ 167.0°); \\ \Delta_4^{(1-4)} &= (131.1°,\ 53.4°,\ 1.3°,\ 159.7°\ ); \end{aligned} \tag{76'}$$

Fig.8 shows the dependence of condition number on orientations of LC transducers α_3, α_4.

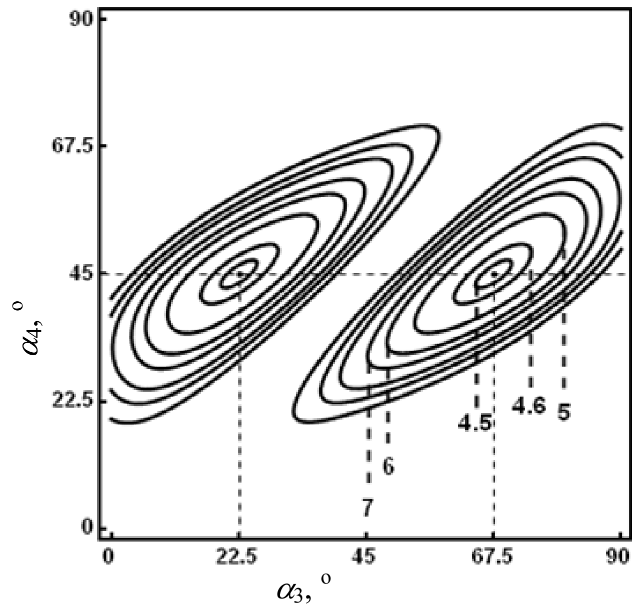


Fig.8. Dependence of condition number on orientations of LC transducers α_3, α_4.

After determination of actual range of phase shift changing for LC transducers ($\Delta_{3,4} \in [30^o, 180^o]$), we corrected the conditions for optimal determination of Stokes parameters by polarimeter Fig.7. As a result the optimal set of values of phase shifts of LC transducers in comparison with previous estimations has changed. New (corrected) set is

$$\left.\begin{aligned} \Delta_3^{(1-4)} &= \left(40,\ \ 40,\ \ 180,\ \ 180°\right); \\ \Delta_4^{(1-4)} &= \left(40,\ \ 138.52,\ \ 173.26,\ \ 59.90\ \ \right) \end{aligned}\right\}; \tag{77}$$

At the same time, the optimum orientation of LC transducers remains the same as previous one Eq.(76)

When the optimum set of parameters of transducers was established, we analyze how the imperfections of these transducers and other elements of polarimeter affect an accuracy of each of the four Stokes parameters. To estimate errors for each Stokes parameters we used the known relation for the errors of indirect measurements [31]:

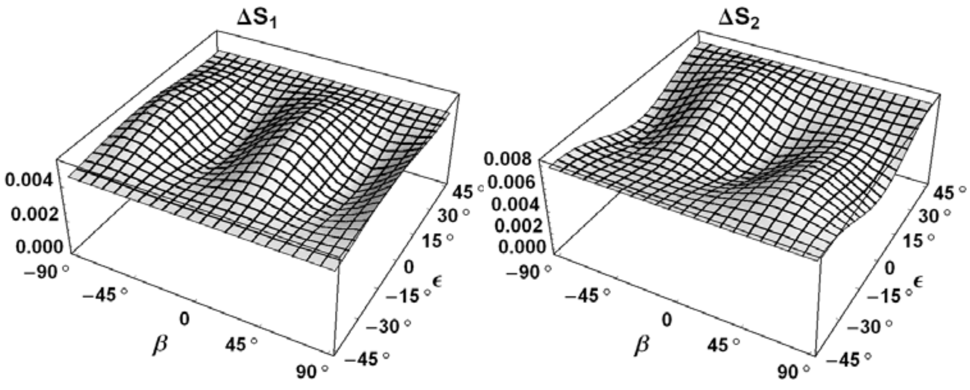
$$\delta F(y_1, y_2, ..., y_N) = \sqrt{\sum_{n=1}^{N}\left(\frac{\partial F(y_n^0)}{\partial y_n}\delta y_n\right)^2}\ , \tag{78}$$

where $F(y_1, y_2, ..., y_N)$ designates some function of measured variables y_n; y_n^0 - mathematical expectation (averaged value).

According to Eqs.(72) and (78) the expression for the estimations of Stokes parameter's errors take the form:

$$\begin{aligned} \Delta S_i = &\left[\sum_{k=1}^{4}\left(\left(\frac{\partial S_i}{\partial \Delta_1^{(k)}}\right)^2 + \left(\frac{\partial S_i}{\partial \Delta_2^{(k)}}\right)^2 \Delta\Delta\right)\right. \\ &\left. + \left(\frac{\partial S_i}{\partial \alpha_1}\Delta\alpha\right)^2 + \left(\frac{\partial S_i}{\partial \alpha_2}\Delta\alpha\right)^2 + \left(\frac{\partial S_i}{\partial I}\Delta I\right)^2\right]^{1/2} \end{aligned}. \tag{79}$$

To obtain quantitative estimations for expression (79) we set the next values of errors: $\Delta\alpha$=0.2°, $\Delta\Delta$=0.5° and ΔI=0.1% (taking the intensity as I = 1). As a result, basing on Eq.(79), we obtain the following dependences of errors for each of Stokes parameters ΔS_i on azimuth β and ellipticity angle ε of analysed polarization ellipse (Fig.9).



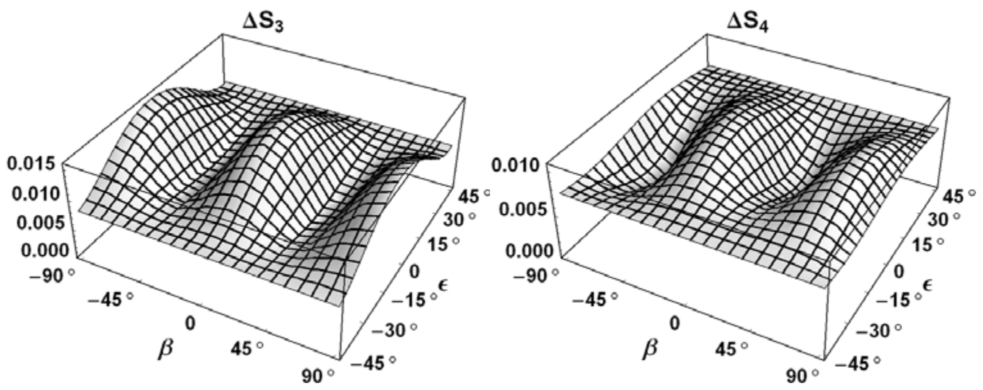


Fig.9. Dependences of Stokes parameter's errors on polarization of analysed radiation (β - azimuth of polarization ellipse, ε - ellipticity angle).

Mean $\overline{\Delta \mathbf{S}}$ and peak $\overline{\Delta \mathbf{S}}_{pp}$ value of error for each Stokes parameter is:

$$\begin{aligned} \overline{\Delta \mathbf{S}} &= \begin{bmatrix} 0.005 & 0.007 & 0.010 & 0.008 \end{bmatrix} \\ \Delta \mathbf{S}_{pp} &= \begin{bmatrix} 0.001 & 0.001 & 0.006 & 0.003 \end{bmatrix} \end{aligned} \tag{80}$$

As it is for Stokes-polarimeter the optimal polarizations $\mathbf{R}^{(1\text{-}4)}$ of input radiation are generated for the next pairs of values of phase shifts of LC transducers 1 and 2:

$$\begin{aligned} \Delta_1^{(1-4)} &= \left(40, 40, 180, 180°\right); \\ \Delta_2^{(1-4)} &= \left(40, 138.52, 173.26, 59.90\ \right); \end{aligned} \tag{81}$$

This result was obtained by minimisation of condition number of the characteristic matrix **G** Eq.(70). The expressions for the Stokes parameters of input radiation are:

$$\mathbf{R}^{(i)} = \mathbf{M}_{LCVR-2}(\Delta_2^{(i)}, \alpha_2) \cdot \mathbf{M}_{LCVR-1}(\Delta_1^{(i)}, \alpha_1) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{82}$$

In accordance with scheme Fig.6 we have assembled the dummy of Mueller-polarimeter and have developed software modules to control of its operating. Improved software interface is shown below in Fig.10.

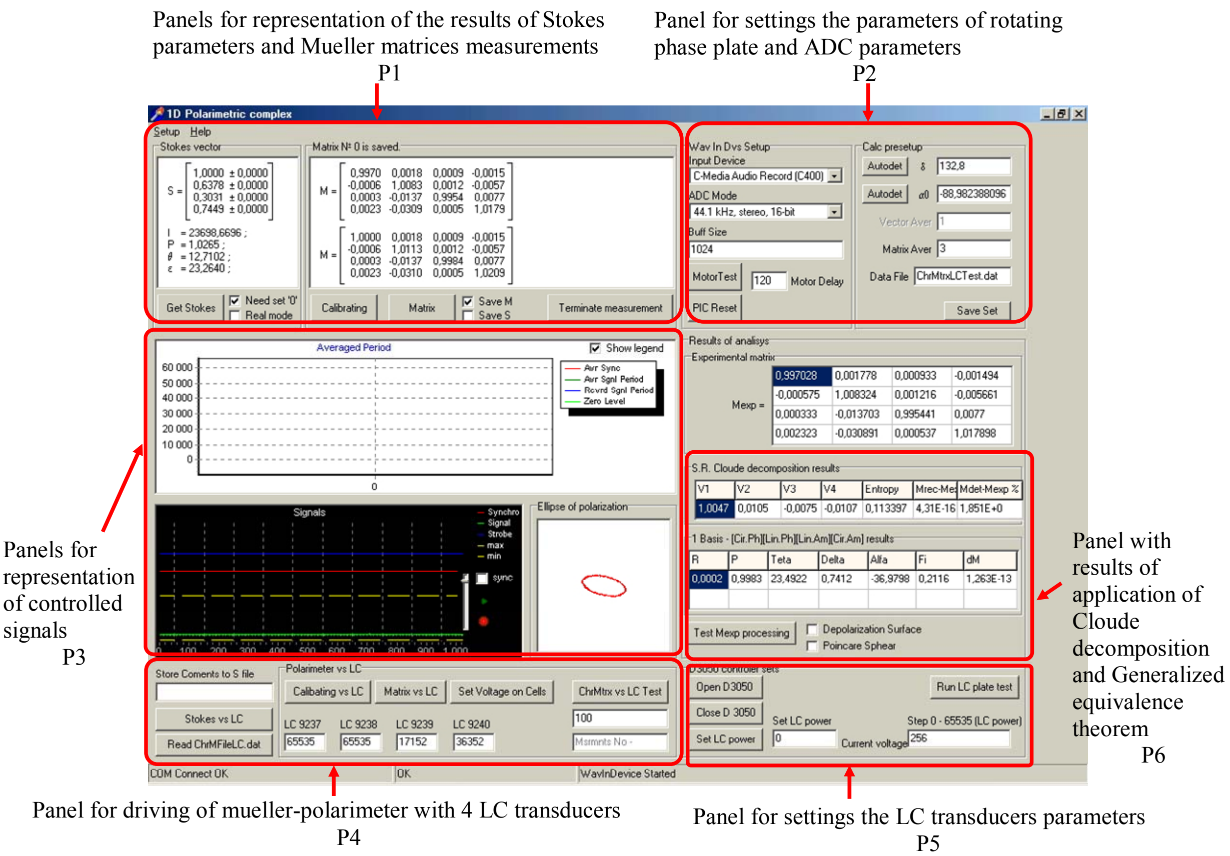


Fig.10. Developed software for Mueller matrix polarimeter operation and control which realizes the time-sequential strategy with using four LC transducers. Modules with Cloude's decomposition and generalized equivalence theorem are implemented as well (see panel P6).

Measurement procedure with developed polarimeter consists of the next sequence of operations:

1. First, we need to define polarizations of probing radiation. To do this we need to remove all objects from probing channel of polarimeter (measurement mode "without object") and press the button "Calibrating vs LC" of developed software which placed on driving panel P4 (see Fig.10). Then, one has to follow to options of corresponding dialog menu. As a result input Stokes vectors are formed and represented on the right of panel P1.

2. Second, to measure the Mueller matrix we insert an investigated object on the way of laser beam after PSG (see Fig.6) and press the button "Matrix vs LC" and follow to the options of corresponding dialog menu again. Measured matrix will be figured on panel P1 as M: (nonnormalized one is on the top and normalized by m_{11} on the bottom).

3. If we want to determine the parameters of polarization (θ - azimuth, ε - ellipticity, I - intensity, P – polarization degree, S - Stokes parameters) which measured by PSA (Fig.6) we have just press the button "Stokes vs LC" and wait until required information will be figured on the corresponding panel P1 (see Fig.10).

3. Results and Discussion

Let verify the derived conditions of eigenpolarizations orthogonality for some simple cases. We will analyze the orthogonality conditions in terms of the Mueller matrix elements Eq.(31), Eq.(42) (condition Eq.(28) is omitted because it is a particular case of Eq.(42)).

As it was pointed out previously, the media characterized by only one of the basic types of anisotropy possess always orthogonal eigenpolarizations. Mueller matrices of these types of anisotropy are given in Eq.(12). As can be directly seen, for all of them the condition Eq.(41) is always satisfied. Conditions Eq.(31) in these cases of anisotropy become an identity or have the uncertainty of the type 0/0.

Multiple numerical verifications show that if the model Eq.(10) consists all anisotropy parameters, conditions Eq.(31) and Eq.(42) are satisfied if values of circular and linear anisotropy satisfy Eq.(55), Eq.(56). For example:

$$P = 0.2;\quad \gamma = 25^o;\quad R = 0.7;\quad \alpha = 84^o;\quad \Rightarrow\quad \varphi = 9.38^o;\quad \Delta = -51.83^o. \tag{83}$$

$$\mathbf{T} = \mathbf{T}^{CP}\mathbf{T}^{lP}\mathbf{T}^{CA}\mathbf{T}^{LA} = \begin{bmatrix} 0.803 + 0.61i & 0.457 + 0.098i \\ 0.188 + 0.428i & 0.261 + 0.168i \end{bmatrix} \tag{84}$$

It is easy to see that eigenpolarizations $\chi_{1,2}$ of the Jones matrix Eq.(84) are orthogonal. Indeed:

$$\begin{gathered} \chi_1 = 0.446 + 0.228i;\ \chi_2 = -1.777 \text{ - } 0.908i; \\ \chi_1\chi_2^* = -1. \end{gathered} \tag{85}$$

Taking to account Eq.(83), in terms of Mueller matrix we obtain:

$$\mathbf{M} = \mathbf{M}^{CP}\mathbf{M}^{lP}\mathbf{M}^{CA}\mathbf{M}^{LA} = \begin{bmatrix} 1.000 & 0.593 & 0.707 & 0.361 \\ 0.593 & 0.436 & 0.394 & 0.155 \\ 0.707 & 0.356 & 0.568 & 0.260 \\ 0.361 & 0.229 & 0.197 & 0.238 \end{bmatrix} \tag{86}$$

Eigenpolarizations of the Mueller matrix $\mathbf{M}$ Eq.(86)

$$\mathbf{S}_1 = \begin{bmatrix} 1.000 \\ 0.599 \\ 0.713 \\ 0.365 \end{bmatrix};\quad \mathbf{S}_2 = \begin{bmatrix} 1.000 \\ -0.599 \\ -0.713 \\ -0.365 \end{bmatrix};\quad \mathbf{S}_1 + \mathbf{S}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{87}$$

При этом Eq.(31) $\Rightarrow$ 0.034=0.034=0.034

In order to check the orthogonality conditions in terms of anisotropy parameters Eq.(52)-Eq.(54) let us consider some special cases.

(**i**) In case of absence of amplitude anisotropy ($P = 1$ and $R = 0$) we have:

$$\begin{gathered} 0 = 0 \\ 0 = 0. \\ 0 = 0 \end{gathered} \tag{88}$$

The Jones matrix for the case of $P = 1$ and $R = 0$ is a unitary [2] and, indeed, characterized by orthogonal eigenpolarizations.

(**ii**) In the case of absence of phase anisotropy ($\Delta = 0$ and $\varphi = 0$), Eq.(28) takes the form:

$$\begin{aligned} &(1-P)^2 R = 0 \\ &2(1-P)R^2\cos 2\gamma = 0 \\ &(1-P^2)R^2\sin 2\gamma = 0 \end{aligned} \tag{89}$$

This proves again that simultaneous presence of both type of amplitude anisotropy results in non-orthogonality of eigenpolarizations [1].

It is important to note that relations Eq.(52)-Eq.(56) can be considered as a set of rules for synthesis of the polarization elements with orthogonal eigenpolarizations as well.

Results obtained in section 2.1 show that Mueller matrix of the medium with orthogonal eigenpolarizations has characteristic symmetry, at least such as in Eq.(42). At the same time we show that such class of media can generally contain all types of anisotropy.

Next important result is that the medium with orthogonal eigenpolarizations have no more than 4 degrees of freedom. Thus, we can claim that the knowledge of incomplete Mueller matrices Eq.(90) are sufficient for complete description of anisotropy of the medium with orthogonal eigenpolarizations:

$$\begin{bmatrix} m_{11} & * & * & * \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}, \quad \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ * & m_{22} & m_{23} & m_{24} \\ * & m_{32} & m_{33} & m_{34} \\ * & m_{42} & m_{43} & m_{44} \end{bmatrix}. \tag{90}$$

Furthermore, taking into account the relations [34] the number of matrix elements that need to be measured may be further decreased.

Fig.4 shows that LC transducers in addition to linear birefringence has an evident linear amplitude anisotropy depending on applied voltage as well. This needs to be taken into account during polarimeter adjustments. In particular it is reasonable to operate with applied voltage more then 3V.

It was shown in [14] that minimum value for condition number of instrumental matrix with dimension 4x4 for time-sequential Stokes-polarimeter with rotating phase plate which can principally be obtained is $V_{\min} = 4.47$. For the case of Stokes-polarimeter with two fixed LC transducers we obtain value $V_{\min} = 4.50$ that is not far from principally obtainable minimum. This allows providing minimal errors in measurement of Stokes parameters. Note, these results concerns so-called integral errors of Stokes vector measurement Eqs.(74) and Eq.(75).

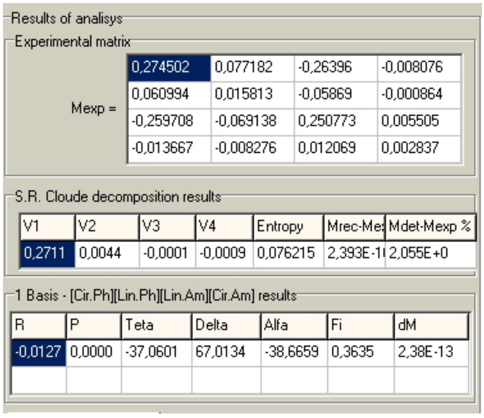
Unfortunately, in practice an actual range of change of phase shift for LC transducers lies in region $\Delta \in [30^{\circ}, 180^{\circ}]$ (smooth regions on dependencies presented in Fig.4). So, we have found new set of $\Delta_{1,2,3,4}$ Eq.(77). This results in some increasing of condition number $V_{\min}$ (from 4.5 to 4.6). However, under equal other conditions this increases the limits of relative error $\delta\mathbf{S}$ just up to 1.02 times.

As it can be seen from Fig.9, the error of Stokes parameters determination by polarimeter Fig.7 depends on the polarization of input radiation. This dependence is symmetrical relative to sign of ellipticity of polarization ellipse for parameters s_1 and s_3, and is asymmetric for parameters s_2 and s_4. All dependences in Fig.9, except for parameter s_3, are asymmetric relative to zero value of azimuth β. From Eq.(80) it can also be concluded that in the scheme of polarimeter, which is considered, the Stokes parameters are generally measured with different accuracy. Meanwhile parameter s_1 is most precisely defined, s_2 a little worse, than S_4 and parameter s_3 defined with lowest precision. For the parameter s_3 the magnitude of error is also highest. The positions of extremum of dependences Fig.9 are determined by LC transducers' axes orientation ($\alpha_1 = 22.5°$, $\alpha_2 = 45°$).

Thus, in considered polarimeter scheme for arbitrary polarization of radiation, intensity will be determined with maximum accuracy. Polarization parameter - large axis of the polarization ellipse that oriented at 0° and 90°, will also be determined with the greatest accuracy for arbitrary ellipticity. Accuracy of analysis of the polarization ellipse orientation close to 45° is

the worst, and strongly depends on ellipticity of ellipse. Ellipticity angle of polarization ellipse in described scheme of polarimeter, taking into account comparability of values $\overline{\square s_4}$ and $\square s_{pp,4}$ is determined with greater error relative to azimuth determination and depends on the orientation of the polarization ellipse.

The algorithms of Cloude decomposition and decomposition in accordance with generalised equivalence theorem described in section 2.2 are implemented in appropriate software module Fig.10. It provides additional information about studied objects. In particular, in the case shown in Fig.10, it can be seen that measurement of Mueller matrix was carried out in the mode "without object." In this case the measured Mueller matrix is unitary diagonal within error range $\delta\mathbf{M} \approx 1.9\%$. This do result in relatively large entropy H=0.11. Values of anisotropy parameters indicate that measured object (empty space) does not characterized by any type of anisotropy just as it had expected. Next figure demonstrates the part of interface Fig.10 when prism polarizer is measured:


Results of analisys

Experimental matrix

Mexp =

0,274502	0,077182	-0,26396	-0,008076
0,060994	0,015813	-0,05869	-0,000864
-0,259708	-0,069138	0,250773	0,005505
-0,013667	-0,008276	0,012069	0,002837

S.R. Cloude decomposition results

V1	V2	V3	V4	Entropy	Mrec-Me	Mdet-Mexp %
0,2711	0,0044	-0,0001	-0,0009	0,076215	2,393E-1	2,055E+0

1 Basis - [Cir.Ph][Lin.Ph][Lin.Am][Cir.Am] results

R	P	Teta	Delta	Alfa	Fi	dM
-0,0127	0,0000	-37,0601	67,0134	-38,6659	0,3635	2,38E-13

Fig.11. Part of interface Fig.10 demonstrating the case when measured object is a prism polarizer.

Fig.11 represents the Mueller matrix with structure typical for linear amplitude anisotropy: entropy H is about 0.08; main anisotropy type (linear amplitude anisotropy with value "P"=0, and azimuth "Teta" = -37^{o}.

Analysis of the Muller matrix of structure Eq.(90) has shown that its elements can not be determined independently by means of time-sequential approach described above, see Eq.(70). Since the first row of the matrix is responsible for converting the intensity of the radiation by medium, they will be involved in the construction of relations Eq.(70) [37,38].

From the other hand, for determining of elements of the Mueller matrix there is no need to use the Stokes polarimeter in receiving channel of polarimeter. System of simultaneous equations relative to 13 matrix elements of the form Eq.(90) can be composed basing on relations Eq.(72) directly:

$$\begin{aligned} I_k &= \mathbf{R}(\theta_A, \Delta_4^{(k)}, \Delta_3^{(k)}, \alpha_4, \alpha_3) \cdot \mathbf{M_{13}} \cdot \mathbf{S}(\Delta_2^{(k)}, \Delta_1^{(k)}, \alpha_2, \alpha_1, \theta_P) = \\ &= r_1 s_1 m_{11} + (r_1 s_2 + r_2 s_1) m_{12} + (r_1 s_3 + r_3 s_1) m_{13} + (r_1 s_4 + r_4 s_1) m_{14} + \\ &\quad + r_2 s_2 m_{22} + r_2 s_3 m_{23} + r_2 s_4 m_{24} + \\ &\quad + r_3 s_2 m_{32} + r_3 s_3 m_{33} + r_3 s_4 m_{34} + \\ &\quad + r_4 s_2 m_{42} + r_4 s_3 m_{43} + r_4 s_4 m_{44}, \end{aligned} \tag{91}$$

where k=1-13, I_k – measured intensity in the case of k-th set of phase shifts of LC transducers and

$$\mathbf{R}(\theta_A,\Delta_4^{(k)},\Delta_3^{(k)},\alpha_4,\alpha_3)=\left[\mathbf{M}_A(\theta_A)\cdot\mathbf{M}_{LCVR-4}(\Delta_4^{(k)},\alpha_4)\cdot\mathbf{M}_{LCVR-3}(\Delta_3^{(k)},\alpha_3)\right]^{<1>},$$

$$\mathbf{S}(\theta_P,\Delta_2^{(k)},\Delta_1^{(k)},\alpha_2,\alpha_1)=\mathbf{M}_{LCVR-2}(\Delta_2^{(k)},\alpha_2)\cdot\mathbf{M}_{LCVR-1}(\Delta_1^{(k)},\alpha_1)\cdot\begin{bmatrix}1\\ \cos(2\theta_P)\\ \sin(2\theta_P)\\ 0\end{bmatrix}, \tag{92}$$

<1> - denotes first row of resulting matrix and $\mathbf{M_{13}}$ denotes the following Mueller matrix:

$$\mathbf{M_{13}}=\begin{bmatrix}m_{11} & m_{12} & m_{13} & m_{14}\\ m_{12} & m_{22} & m_{23} & m_{24}\\ m_{13} & m_{32} & m_{33} & m_{34}\\ m_{14} & m_{42} & m_{43} & m_{44}\end{bmatrix}. \tag{93}$$

From Eq.(91)- Eq.(93) we can form the system of equations:

$$\underbrace{\left[I_k\right]}_{\mathbf{I}}=\underbrace{\begin{bmatrix}r_1^k s_1^k & r_1^k s_2^k+r_2^k s_1^k & r_1^k s_3^k+r_3^k s_1^k & r_1^k s_4^k+r_4^k s_1^k & & & & & \\ & & & r_2^k s_2^k & r_2^k s_3^k & r_2^k s_4^k & r_3^k s_2^k & r_3^k s_3^k & \\ & & & & & r_3^k s_4^k & r_4^k s_2^k & r_4^k s_3^k & r_4^k s_4^k\end{bmatrix}}_{\mathbf{B_{13x13}}}\overrightarrow{\mathbf{M}}_{\mathbf{13}};\quad k=\overline{1-13}. \tag{94}$$

where $\mathbf{I}$ – vector of intensities, $\mathbf{B_{13x13}}$ – characteristic matrix of dimension 13x13; $\overrightarrow{\mathbf{M}}_{\mathbf{13}}$ - the Mueller vector of the form:

$$\overrightarrow{\mathbf{M}}_{\mathbf{13}}=\left[m_{11}\quad m_{12}\quad m_{13}\quad m_{14}\quad m_{22}\quad m_{23}\quad m_{24}\quad m_{32}\quad m_{33}\quad m_{34}\quad m_{42}\quad m_{43}\quad m_{44}\right]^T. \tag{95}$$

From Eq.(94) we have for m_{ij} :

$$\overrightarrow{\mathbf{M}}_{\mathbf{13}}=\left(\mathbf{B_{13x13}}\right)^{-1}\mathbf{I}. \tag{96}$$

As it was in the case of Stokes-polarimeter, see Eqs.(72)-(75), to minimize an error of Mueller matrix elements determination in Eq.(94) we have minimize the condition number $V_{B_{13x13}}$ of characteristic matrix $\mathbf{B_{13x13}}$. From Eq.(94) it can be seen that $V_{B_{13x13}}$ is a function of 10 parameters $(\theta_A,\theta_P,\alpha_{1-4},\Delta_{1-4})$. It is obvious that we can set $\theta_P=0$ and other azimuths are determined relative to θ_P. Then $V_{B_{13x13}}$ was numerically minimized taking into account that actual range of phase shift changing for LC transducers is $\Delta_{1\text{-}4}\in[30^{\circ},180^{\circ}]$. As a result the minimum value $V_{B_{13x13},\min}\approx 20.23$ under next conditions:

$$\begin{bmatrix}\Delta_1^{(1-13)}\\ \Delta_2^{(1-13)}\\ \Delta_3^{(1-13)}\\ \Delta_4^{(1-13)}\end{bmatrix}=\begin{bmatrix}42.4^\circ & 91.5^\circ & 122.8^\circ & 40.0^\circ & 135.4^\circ & 180^\circ & 120.4^\circ & 40.9^\circ & 180^\circ & 177.8^\circ & 73.1^\circ & 40.5^\circ & 84.5^\circ\\ 54.7^\circ & 179.8^\circ & 174.3^\circ & 177.4^\circ & 179.9^\circ & 54.8^\circ & 145.0^\circ & 178.5^\circ & 40.1^\circ & 41.0^\circ & 44.6^\circ & 173.8^\circ & 65.6^\circ\\ 95.9^\circ & 40.1^\circ & 112.5^\circ & 94.9^\circ & 180^\circ & 179.6^\circ & 179.7^\circ & 156.7^\circ & 144.6^\circ & 47.2^\circ & 40.3^\circ & 180^\circ & 178.8^\circ\\ 165^\circ & 88.5^\circ & 138.8^\circ & 144.6^\circ & 40.3^\circ & 178.1^\circ & 179.2^\circ & 40.1^\circ & 51.5^\circ & 113.0^\circ & 81.4^\circ & 179.8^\circ & 42.8^\circ\end{bmatrix} \tag{97}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}=\begin{bmatrix}-133.1^\circ\\ -69.3^\circ\\ 99.2^\circ\\ 62.6^\circ\end{bmatrix},\ \theta_A=0^\circ .$$

It is important to note that in the case of determination of all 16 elements of Mueller matrix by approach Eq.(91)- Eq.(96) (i.e., k=16), the absolute minimum of condition number is $V_{B_{16x16},\min}=20\approx 4.47^2$. It is reached for condition Eq.(76) rewritten in form:

$$\begin{bmatrix}\Delta_1^{(1-16)}\\ \Delta_2^{(1-16)}\\ \Delta_3^{(1-16)}\\ \Delta_4^{(1-16)}\end{bmatrix}=\begin{bmatrix}3.2^\circ & 3.2^\circ & 3.2^\circ & 3.2^\circ & 162.5^\circ & 162.5^\circ & 162.5^\circ & 162.5^\circ & 32.6^\circ & 32.6^\circ & 32.6^\circ & 32.6^\circ & 167.0^\circ & 167.0^\circ & 167.0^\circ & 167.0^\circ\\ 131.1^\circ & 131.1^\circ & 131.1^\circ & 131.1^\circ & 53.4^\circ & 53.4^\circ & 53.4^\circ & 53.4^\circ & 1.3^\circ & 1.3^\circ & 1.3^\circ & 1.3^\circ & 159.7^\circ & 159.7^\circ & 159.7^\circ & 159.7^\circ\\ 131.1^\circ & 162.5^\circ & 32.6^\circ & 167.0^\circ & 131.1^\circ & 162.5^\circ & 32.6^\circ & 167.0^\circ & 131.1^\circ & 162.5^\circ & 32.6^\circ & 167.0^\circ & 131.1^\circ & 162.5^\circ & 32.6^\circ & 167.0^\circ\\ 3.2^\circ & 53.4^\circ & 1.3^\circ & 159.7^\circ & 3.2^\circ & 53.4^\circ & 1.3^\circ & 159.7^\circ & 3.2^\circ & 53.4^\circ & 1.3^\circ & 159.7^\circ & 3.2^\circ & 53.4^\circ & 1.3^\circ & 159.7^\circ\end{bmatrix},\tag{98}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}=\begin{bmatrix}22.5^\circ\\ 45^\circ\\ 45^\circ\\ 22.5^\circ\end{bmatrix}\ \theta_A=0^\circ .$$

Taking into account that for LC transducers $\Delta_{1\text{-}4}\in[30^o,180^o]$ we get $V_{B_{16x16},\min}{}^{*}=20.62$ for conditions Eq.(77) and Eq.(81) rewritten in form Eq.(98). Thus, 13 elements of the Mueller matrix Eq.(93) with orthogonal eigenpolarizations can be determined from system of equations Eq.(94) more precisely than complete matrix in general case since $V_{B_{16x16},\min}{}^{*}>V_{B_{13x13},\min}$.

For time-sequential approach Eqs.(94) and (96) the developed software Fig.10 improves the measuring of complete Mueller matrix and incomplete Mueller matrix of the form Eq.(93) by breadboard of Mueller-polarimeter, Fig.6, with optimal parameters Eq.(97) and Eq.(98).

Before testing the developed polarimeter, Fig.6, we performed a series of calibrating measurements by polarimeter Fig. 5 to determine of real parameters of LC transducers (their Mueller matrices) when driving voltage had been applied in accordance with Eq.(97) and Eq.(98). Then elements of vectors **R, S** and matrix $\mathbf{B_{nxn}}$ had been calculated in accordance with Eq.(92). Here **n** is equal to 16 for measurement of complete Mueller matrix and 13 for matrix of the structure Eq.(90), correspondingly.

Finally, by calibrated polarimeter we had performed measurements of the Mueller matrices of linear polarizer (prism Glana) and polymer achromatic quarter wave plate with orientations 0^o, 45^o, 90^o. Measurements were carried out with 13 intensities in the case of incomplete matrix Eq.(93) and with 16 intensities in the case of complete Mueller matrix. Results of these measurements and error assessments are shown below in the tables 1 and 2. For comparison table 3 contains tabular Mueller matrices of 91^o wave plate and linear polarizer with P=0 with orientations 0^o, 45^o, 90^o.

Table 1. Results of measurements of Mueller matrices of achromatic zero order quarter wave plate ($\Delta\approx 91^o$) and linear polarizer ($P\approx 0$) by 13 intensities approach.

quarter wave plate

$\alpha=0^o$ (δM=1.4%) $\alpha=45^o$(δM=1.4%) $\alpha=90^o$(δM=1.4%)

$$\begin{pmatrix} 1.0155 & 0.0008 & 0.0001 & -0.0014 \\ 0.0008 & 1.0139 & 0.0012 & 0.0001 \\ 0.0001 & -0.0003 & -0.0177 & 1.0132 \\ -0.0014 & -0.0002 & -1.0138 & -0.0174 \end{pmatrix} \begin{pmatrix} 1.0157 & -0.0010 & -0.0005 & 0.0000 \\ -0.0010 & -0.0159 & 0.0010 & -1.0139 \\ -0.0005 & 0.0002 & 1.0121 & -0.0008 \\ 0.0000 & 1.0140 & 0.0007 & -0.0186 \end{pmatrix} \begin{pmatrix} 1.0149 & 0.0011 & 0.0004 & 0.0001 \\ 0.0011 & 1.0141 & 0.0018 & -0.0011 \\ 0.0004 & 0.0003 & -0.0180 & -1.0138 \\ 0.0001 & -0.0015 & 1.0131 & -0.0196 \end{pmatrix}$$

linear polarizer

$\gamma=0^o$ (δM=1.5%) $\gamma=45^o$ (δM=1.4%) $\gamma=90^o$ (δM=1.4%)

$$\begin{pmatrix} 1.0155 & 1.0145 & -0.0008 & -0.0005 \\ 1.0145 & 1.0155 & 0.0013 & 0.0028 \\ -0.0008 & 0.0002 & 0.0001 & 0.0005 \\ -0.0005 & -0.0006 & -0.0004 & 0.0015 \end{pmatrix} \begin{pmatrix} 1.0157 & -0.0003 & 1.0146 & 0.0003 \\ -0.0003 & 0.0015 & 0.0008 & -0.0018 \\ 1.0146 & -0.0006 & 1.0127 & -0.0009 \\ 0.0003 & 0.0009 & -0.0006 & 0.0001 \end{pmatrix} \begin{pmatrix} 1.0147 & -1.0140 & 0.0003 & 0.0001 \\ -1.0140 & 1.0131 & -0.0011 & -0.0001 \\ 0.0003 & 0.0008 & 0.0016 & 0.0009 \\ 0.0001 & -0.0006 & 0.0010 & 0.0001 \end{pmatrix}$$

Table 2. Results of measurements of Mueller matrices of achromatic zero order quarter wave plate ($\Delta\approx91^o$) and pure linear polarizer ($P\approx0$) by 16 intensities approach.

quarter wave plate

$\alpha=0^o$ (δM=1.4%) $\alpha=45^o$(δM=1.4%) $\alpha=90^o$(δM=1.4%)

$$\begin{pmatrix} 1.0154 & -0.0006 & 0.0007 & -0.0004 \\ 0.0007 & 1.0144 & 0.0018 & 0.0004 \\ -0.0001 & -0.0017 & -0.0181 & 1.0129 \\ -0.0001 & 0.0004 & -1.0145 & -0.0175 \end{pmatrix} \begin{pmatrix} 1.0162 & 0.0008 & 0.0000 & -0.0005 \\ -0.0001 & -0.0188 & 0.0006 & -1.0133 \\ 0.0006 & 0.0008 & 1.0112 & 0.0007 \\ 0.0004 & 1.0133 & 0.0004 & -0.0173 \end{pmatrix} \begin{pmatrix} 1.0149 & -0.0003 & 0.0003 & 0.0007 \\ -0.0010 & 1.0153 & -0.0007 & 0.0005 \\ -0.0007 & 0.0016 & -0.0169 & -1.0134 \\ -0.0002 & -0.0005 & 1.0124 & -0.0177 \end{pmatrix}$$

linear polarizer

$\gamma=0^o$ (δM=1.5%) $\gamma=45^o$ (δM=1.4%) $\gamma=90^o$ (δM=1.4%)

$$\begin{pmatrix} 1.0151 & 1.0145 & -0.0009 & 0.0018 \\ 1.0149 & 1.0140 & -0.0010 & 0.0007 \\ -0.0016 & -0.0015 & 0.0004 & 0.0021 \\ -0.0010 & 0.0004 & -0.0049 & -0.0014 \end{pmatrix} \begin{pmatrix} 1.0144 & 0.0002 & 1.0142 & -0.0003 \\ -0.0001 & 0.0005 & -0.0010 & 0.0002 \\ 1.0141 & -0.0001 & 1.0140 & 0.0002 \\ -0.0006 & 0.0009 & -0.0013 & -0.0005 \end{pmatrix} \begin{pmatrix} 1.0154 & -1.0129 & 0.0017 & -0.0011 \\ -1.0149 & 1.0117 & -0.0007 & -0.0007 \\ 0.0020 & 0.0012 & 0.0038 & -0.0019 \\ -0.0007 & 0.0000 & -0.0014 & -0.0003 \end{pmatrix}$$

Table 3. Tabular Mueller matrices of 91^o wave plate and linear polarizer with P=0 with orientations 0^o, 45^o, 90^o.

wave plate

$\alpha=0^o$ $\alpha=45^o$ $\alpha=90^o$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.017 & 1 \\ 0 & 0 & -1 & -0.017 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.017 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -0.017 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.017 & -1 \\ 0 & 0 & 1 & -0.017 \end{pmatrix}$$

linear polarizer

$\gamma=0^o$ $\gamma=45^o$ $\gamma=90^o$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Error of measurements described in the tables above was calculated as

$$\delta\mathbf{M} = \frac{\left\|\mathbf{M}_{\mathrm{tab}} - \mathbf{M}_{\mathrm{exper}}\right\|}{\left\|\mathbf{M}_{\mathrm{tab}}\right\|} \cdot 100\%, \qquad (99)$$

where $\mathbf{M}_{\mathrm{tab}}$ – tabular (exact) Mueller matrices calculated for studied objects with given orientations; $\mathbf{M}_{\mathrm{exper}}$ – measured matrices. $\| \ \|$ - Euclidian norm.

All matrices in tables 2 and 3 are nonnormalized and are results of 600 times averaging. The time of measurement is about 5 sec for complete Mueller matrix and about 4 sec for incomplete Mueller matrix Eq.(93).

Taking into account the fact that the number of informative matrix elements can be less that 13, see for example [33,34], the increasing of the accuracy which is achieved by proposed method can potentially be higher (several times or even order). Indeed, let us consider the following cases.

First: the Mueller matrix describing the objects characterized by elliptical phase anisotropy and possibly isotropic depolarization. These can be the media of biological nature, see, for example, [8] and references herein. The matrix has a form:

$$\mathbf{M_{10}} = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix} \qquad (100)$$

The analysis shows that minimum value of condition number for matrix Eq.(100) is 13.2 or by one and a half times lesser that one for complete Mueller matrix. It is reached for conditions

$$\begin{bmatrix} \Delta_1^{(1-10)} \\ \Delta_2^{(1-10)} \\ \Delta_3^{(1-10)} \\ \Delta_4^{(1-10)} \end{bmatrix} = \begin{bmatrix} 125.5° & 175.1° & 141.9° & 153.2° & 60.4° & 179.5° & 52° & 57.4° & 51.6° & 57.4° \\ 160.9° & 112.8° & 166.2° & 171.3° & 178.6° & 49.1° & 43.3° & 52.3° & 179.9° & 46.2° \\ 40.6° & 177.4° & 179.8° & 92.1° & 46° & 111.4° & 169.7° & 72.4° & 175.3° & 63° \\ 50.8° & 174.3° & 51.2° & 120.9° & 150.4° & 78.9° & 65.6° & 161.1° & 90.1° & 71.9° \end{bmatrix}$$

(101)

and:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 37.1° \\ 98.7° \\ 90.6° \\ -53.1° \end{bmatrix}. \qquad (102)$$

Second example is

$$\mathbf{M}_6 = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & m_{44} \end{bmatrix} \qquad (103)$$

This matrix describes the media characterized by circular phase anisotropy (optical activity) possibly with isotropic depolarization as well. The minimum value of condition number for matrix Eq.(103) is 7.5 and, thus, the error of measurement of the matrix Eq.(103) decreases up to 2.7 times comparing with the case of complete Mueller matrix measurements.

The conditions for the measurements are follows :

$$\begin{bmatrix} \Delta_1^{(1-6)} \\ \Delta_2^{(1-6)} \\ \Delta_3^{(1-6)} \\ \Delta_4^{(1-6)} \end{bmatrix} = \begin{bmatrix} 158.7^\circ & 176^\circ & 40.3^\circ & 131.2^\circ & 40^\circ & 100.8^\circ \\ 40^\circ & 178.5^\circ & 178^\circ & 50.1^\circ & 81.3^\circ & 152.2^\circ \\ 57.6^\circ & 75.4^\circ & 72.1^\circ & 105.6^\circ & 174.2^\circ & 179.2^\circ \\ 146.9^\circ & 119.8^\circ & 80.8^\circ & 40.3^\circ & 179.7^\circ & 91.3^\circ \end{bmatrix} \qquad (104)$$

and:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 58^\circ \\ 91.8^\circ \\ 81.4^\circ \\ 45.1^\circ \end{bmatrix} \qquad (105)$$

4. Conclusions

Summarizing obtained above results we can formulate next conclusions.

Anisotropy of homogeneous media with orthogonal eigenpolarizations has been determined. In particular, when we use for modeling such media an equivalence theorem in the form of Eq.(10), eigenpolarizations will always be orthogonal if linear Δ and circular birefringence φ have the values Eq.(55) and Eq.(56), whereas, other four anisotropy parameters (R, P,γ,α) can take an arbitrary values. One particular case needs to be noted individually: when the medium has simultaneously two types of amplitude anisotropy (circular and linear) only, there no exist any conditions for such medium to have orthogonal eigenpolarizations. This conclusion follows directly from relations Eq.(52)- Eq.(54).

Derived features of anisotropy of media with orthogonal eigenpolarizations leads to special symmetry in respective Jones and Mueller matrices (see Eqs, (14) and (42)) and to additional interrelations between matrix elements (see Eqs.(21), (23), (25), (29), (30), and (31)). These relations can be useful to simplify an analysis of properties of such media and can be utilized for optimization of measurement procedure. In present work we used an equality between elements of first column and row of the Mueller matrix Eq.(42) to reduce the measurement time with preservation of accuracy for polarimeter with four LC transducers. Optimum configurations for transducers are presented in Eq.(97).

Finally, we have assembled and tested breadboard of 1D Mueller-polarimeter with four LC transducers (Fig.6). Developed software for control of the polarimeter operation allows measuring the Mueller matrices basing on 16, 13 and less intensities approaches and analyzing

the measured matrices to obtain depolarization (Cloude's entropy [35, 36]) and anisotropy (in accordance with general equivalence theorem [33]) parameters. Thus, "operational flexibility" of proposed polarimeter which is achieved by using the polarization transducers with four LC and taking into account the exact informativity of the matrix elements, allows considerably increasing the measurement accuracy.

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List of Symbols, Abbreviations, and Acronyms

ϕ_{mn} – phase of Jones matrix elements t_{mn};
$\phi_{i(mn\pm kl)}$ – phase of sum(difference) of Jones matrix elements $i(t_{mn}\pm t_{kl})$;
χ - complex variable;
LA – linear amplitude anisotropy;
CA – circular amplitude anisotropy;
LP – linear phase anisotropy;
CP – circular phase anisotropy;
P – value of linear amplitude anisotropy;
γ – azimuth of linear amplitude anisotropy;
R – value of circular amplitude anisotropy;
Δ – value of phase amplitude anisotropy;
α – azimuth of linear phase anisotropy;
φ – value of circular phase anisotropy;
a, b – real, imagine part of some sums/differences of Jones matrix elements;
LCVR - Liquid Crystal Variable Retarder
E - Jones vector of radiation;
S, R – Stokes vectors;
T, t_{mn} – Jones matrix, Jones matrix elements;
M, m_{mn} - Mueller matrix, Mueller matrix elements;
I – intensity of radiation;
I – vector of intensities;
$\mathbf{S}_0$ –an exact Stokes vector;
$\delta\mathbf{X}$ – relative error of vector **X**;
B – characteristic matrix of stokes-polarimeter;
G – characteristic matrix of Mueller-polarimeter;
V – condition number;
$\mathbf{C}, c_{\mathrm{mn}}$ – coherency matrix;
ψ - eigenvector of coherency matrix;
$\mathbf{M}_D$ - nondepolarizing (deterministic) Muller matricx;
μ_k – eigenvalues of coherence matrix
$\overline{\square\mathbf{S}}$ - mean value of error for each Stokes parameter
$\overline{\square\mathbf{S}}_{pp}$ - peak value of error for each Stokes parameter
H – entropy;
$\theta_{A,(P)}$ = azimuth of analyzer, polarizer
β - azimuth of polarization ellipse
ε - ellipticity angle of polarization ellipse